Computer Exercise TMA026 - 2 Bonus Points

2013-2014

The purpose of this computer exercise is to get acquainted with the FEniCS software by performing a convergence study for an elliptic problem. Consider the following boundary value problem

$$\begin{cases} -\nabla \cdot (a\nabla u) + bu = f & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ -a\frac{\partial u}{\partial n} = u_N & \text{on } \Gamma_N, \\ -a\frac{\partial u}{\partial n} = k(u - u_R) & \text{on } \Gamma_R, \end{cases} \tag{1}$$

for some non-constant functions a, b, f, k and some non-zero functions u_D, u_N, u_R.

Task 1 Create an "exact solution" in the following way. Choose a rather simple domain Ω (rectangle, circle, square) and some simple coefficient functions a, b, k. Then choose a nontrivial function u and compute the corresponding data functions f, u_D, u_N, u_R. That is $f = -\nabla \cdot (a\nabla u) + bu$ and so on.

Task 2 After reading the tutorial available online about elliptic problems [1] and multiple boundary conditions[2], download the FEniCS software[3] and the demos [4] and understand how simpler versions of the problem (1) could be solved using FEniCS, paying particular attention to the difference between single and multiple Neumann/Robin boundary conditions and to how the convergence/error analysis is performed.

Task 3 Compute the finite element solution u_h for the problem (1), for a sequence of meshes, and create a table of the errors $\|u - u_h\|$ and $|u - u_h|_1$ as functions of the mesh-size h. Determine the order of convergence.
This kind of convergence study is not only an illustration of the theory but more importantly it is used as a test of the computer program: a minor programming error can result in a non-optimal convergence rate.

Task 4 If an analytical solution to a problem is not known one can use the solution on the finest mesh as a reference solution in the convergence study. Find a data set (by changing e.g. u_D or a) for which the optimal convergence rate (achieved for smooth data) is not achieved.

Notice that the FEniCS book is freely available online[5]. The study of the first chapter, "A FEniCS tutorial", pages $1 - 75$, is strongly recommended.

[1] http://fenicsproject.org/documentation/tutorial/fundamentals.html
[2] http://fenicsproject.org/documentation/tutorial/materials.html
[3] http://fenicsproject.org/download/
[4] http://fenicsproject.org/pub/book/chapters/01/stationary/poisson/
[5] https://launchpadlibrarian.net/83776282/fenics-book-2011-10-27-final.pdf