Matematik Chalmers

Tentamen i
TMA026 Partiella differentialekvationer, fk, ENM
MMA430 Partiella differentialekvationer II, 2008–05–26 f M

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Inga hjälpmedel. Kalkylator ej tillåten.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

Here $\Omega \subset \mathbf{R}^2$ is a bounded convex domain whose boundary Γ is a polygon.

1. Consider the boundary value problem

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u) &= f(x), && x \in \Omega, \\ a(x)\frac{\partial u}{\partial n} + u &= 0, && x \in \Gamma, \end{aligned}$$

where $0 < a_0 \le a(x) \le a_1$ for all $x \in \Omega$ and n is the outward unit normal on Γ.
(a) Give a weak formulation of this problem and show that it has a unique solution.
(b) Formulate a finite element method for this problem and state and prove an error estimate in the H^1-norm.
Hints: $a(u, v) = (a\nabla u, \nabla v) + (u, v)_\Gamma$ and we know that

$$\|v\|_1 \le C\big(\|\nabla v\| + \|v\|_{L_2(\Gamma)}\big) \quad \forall v \in H^1(\Omega).$$

2. Let u be a solution of the initial-boundary value problem

$$\begin{aligned} u_t - \nabla \cdot (a(x)\nabla u) &= 0, && x \in \Omega,\ t > 0, \\ a(x)\frac{\partial u}{\partial n} + u &= 0, && x \in \Gamma,\ t > 0, \\ u(x, 0) &= v(x), && x \in \Omega, \end{aligned}$$

where a is as in Problem 1. Show that

$$\begin{aligned} \|u(t)\| &\le \|v\|, && t \ge 0, \\ \|u(t)\|_1 &\le C\|v\|_1, && t \ge 0, \\ \|u(t)\|_1 &\le Ct^{-1/2}\|v\|, && t > 0. \end{aligned}$$

3. Consider the initial-boundary value problem for the wave equation

$$\begin{aligned} &u_{tt} - \Delta u = 0, && x \in \Omega,\ t > 0, \\ &u = 0, && x \in \Gamma,\ t > 0, \\ &u(x, 0) = v(x),\ u_t(x, 0) = w(x), && x \in \Omega. \end{aligned}$$

Show the following estimates:

$$\begin{aligned} \|u(t)\|_{H_0^1} + \|u_t(t)\| &\le C\big(\|v\|_{H_0^1} + \|w\|\big), \\ \|u_{tt}(t)\| = \|\Delta u(t)\| &\le C\big(\|\Delta v\| + \|w\|_{H_0^1}\big). \end{aligned}$$

4. State and prove the trace theorem for a square domain.

5. State and prove the maximum principle for the elliptic operator $\mathcal{A}u = -\nabla \cdot (a(x)\nabla u)$. Describe one of its consequences.

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2008-05-26 Solutions

1. a) $(f,v) = -\int_\Omega \nabla\cdot(a\nabla u)v\,dx = -\int_\Gamma \underbrace{n\cdot a\nabla u}_{=-u} v\,dS + (a\nabla u,\nabla v)$

$$= (a\nabla u,\nabla v) + (u,v)_\Gamma$$

$a(u,v) = (a\nabla u,\nabla v) + (u,v)_\Gamma$

$L(v) = (f,v)$

$V = H^1 = H^1(\Omega)$ with norm $\|v\|_1 = \sqrt{\|\nabla v\|^2 + \|v\|^2}$

Weak form: $\begin{cases} u\in H^1 \\ a(u,v) = L(v) \quad \forall v\in H^1 \end{cases}$

$a(v,v) = (a\nabla v,\nabla v) + \|v\|_\Gamma^2 \geq a_0\|\nabla v\|^2 + \|v\|_\Gamma^2$

$\geq \min(a_0,1)\left(\|\nabla v\|^2 + \|v\|_\Gamma^2\right) \geq c\|v\|_1^2$

$|a(u,v)| \leq |(a\nabla u,\nabla v)| + |(u,v)_\Gamma| \leq$

$\leq a_1\|\nabla u\|\,\|\nabla v\| + \|u\|_\Gamma\,\|v\|_\Gamma$

$\left\{\begin{matrix}\text{trace} \\ \text{theorem}\end{matrix}\right\}$ $\leq C\|u\|_1\|v\|_1$

$|L(v)| \leq \|f\|\,\|v\| \leq \|f\|\,\|v\|_1.$

So $a(\cdot,\cdot)$ is an inner product on H^1 equivalent to the standard one.

Riesz repr. theorem: $\exists!$ solution $u \in H^1$.

b) $S_h = \{ v \in C(\bar{\Omega}) : v|_K \in \Pi_1, K \in T_h \}$

$\{T_h\}$ family of triangulations

Then $S_h \subset H^1$.

$$\begin{cases} u_h \in S_h \\ a(u_h, \chi) = L(\chi) \quad \forall \chi \in S_h \end{cases}$$

Error estimates same as in the book.

2. *Weak form*

$$\begin{cases} u(t) \in H^1, \quad u(0) = v \\ (u_t, \varphi) + a(u, \varphi) = 0 \quad \forall \varphi \in H^1, \quad t > 0 \end{cases}$$

with $a(\cdot,\cdot)$ as in Problem 1.

Take $\varphi = u(t)$: $\quad (u_t, u) + \|u\|_a^2 = 0$

$$\tfrac{1}{2} D_t \|u\|^2 + \|u\|_a^2 = 0$$

$$(*) \qquad \|u(t)\|^2 + 2\int_0^t \|u\|_a^2\, ds \le \|v\|^2$$

$$\boxed{\|u(t)\| \le \|v\|}$$

Take $\varphi = u_t(t)$: $\quad \|u_t\|^2 + a(u, u_t) = 0$

$$\|u_t\|^2 + \tfrac{1}{2} D_t \|u\|_a^2 = 0$$

$$2\int_0^t \|u_t\|^2\, ds + \underbrace{\|u(t)\|_a^2}_{\ge c\|u(t)\|_1^2} \le \|v\|_a^2 \le C\|v\|_1^2$$

$$\boxed{\|u(t)\|_1 \le C\|v\|_1}$$

Take $\varphi = t\, u_t(t)$: $\quad t\|u_t\|^2 + t\,\tfrac{1}{2} D_t \|u\|_a^2 = 0$

$$2t\|u_t\|^2 + D_t\left(t\|u\|_a^2\right) = \|u\|_a^2$$

$$t\|u(t)\|_a^2 \le \int_0^t \|u\|_a^2\, ds \overset{(*)}{\le} \tfrac{1}{2}\|v\|_a^2$$

$$\boxed{\|u(t)\|_1 \le C\, t^{-1/2}\|v\|_1}$$

3. $$u(x,t) = \sum_{i=1}^{\infty} \left(\cos(\sqrt{\lambda_i}\, t)\, \hat{v}_i + \frac{1}{\sqrt{\lambda_i}} \sin(\sqrt{\lambda_i}\, t)\, \hat{w}_i \right) \varphi_i(x)$$

$$\|u(t)\|^2_{H_0^1} = \sum \lambda_i\, \hat{u}_i(t)^2 \le 2 \sum \Big(\lambda_i \underbrace{\cos^2(\sqrt{\lambda_i}\, t)}_{\le 1} \hat{v}_i^2 + \underbrace{\sin^2(\sqrt{\lambda_i}\, t)}_{\le 1} \hat{w}_i^2 \Big)$$

$$\le 2 \sum \left(\lambda_i \hat{v}_i^2 + \hat{w}_i^2 \right) = 2 \left(\|v\|^2_{H_0^1} + \|w\|^2 \right)$$

$\|u_t(t)\|^2 =$ the same

$$\|u_{tt}(t)\|^2 \le 2 \sum \left(\lambda_i^2 \cos^2(\sqrt{\lambda_i}\, t)\, \hat{v}_i^2 + \lambda_i \sin^2(\sqrt{\lambda_i}\, t)\, \hat{w}_i^2 \right)$$

$$\le 2 \sum \left(\lambda_i^2 \hat{v}_i^2 + \lambda_i \hat{w}_i^2 \right)$$

$$= 2 \left(\|\Delta v\|^2 + \|w\|^2_{H_0^1} \right)$$

4., 5. see the book.