Matematik Chalmers

Tentamen i
TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2010–05–25 fm V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. For the solution of the homogeneous heat equation

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } \Omega \times \mathbf{R}_+, \\ u &= 0 && \text{on } \Gamma \times \mathbf{R}_+, \\ u(\cdot, 0) &= v && \text{in } \Omega, \end{aligned}$$

we have the bounds

$$\begin{aligned} \int_0^t \|\nabla u(s)\|^2 \, \mathrm{d}s &\le C\|v\|^2, \\ \|\nabla u(t)\|^2 &\le Ct^{-1}\|v\|^2. \end{aligned}$$

(a) Prove these by means of eigenfunction expansion.
(b) Prove these by means of the energy method. Find good values of the constants in both methods.

2. State and prove the maximum principle for the heat equation.

3. Prove the trace inequality for the disk $\Omega = \{x \in \mathbf{R}^2 : |x| < R\}$. Hint: use polar coordinates r, θ so that $x = (x_1, x_2) = (r\cos\theta, r\sin\theta)$ and start from $\displaystyle\int_0^R \frac{\partial}{\partial r}\Big(r\, v(r\cos\theta, r\sin\theta)\Big)^2 \, \mathrm{d}r$.

4. (a) Formulate the finite element method for the elliptic problem:

$$\begin{aligned} -\nabla \cdot \big(a\nabla u\big) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

(b) Formulate sufficient assumptions and prove the error estimate

$$\|u_h - u\| \le Ch^2\|u\|_2.$$

5. The Cahn-Hilliard equation is to find two functions $u = u(x,t)$ and $w = w(x,t)$ such that

$$\begin{aligned} & u_t - \Delta w = 0 && \text{in } \Omega \times \mathbf{R}_+, \\ & w = -\Delta u + f(u) && \text{in } \Omega \times \mathbf{R}_+, \\ & \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 && \text{on } \Gamma \times \mathbf{R}_+, \\ & u(\cdot, 0) = u_0 && \text{in } \Omega. \end{aligned}$$

Here $f(s) = s^3 - s$.
(a) Write down a weak formulation of this problem.
(b) Show that the average of $u(\cdot, t)$ is constant, that is, $\int_\Omega u(\cdot, t)\,\mathrm{d}x = \int_\Omega u_0 \,\mathrm{d}x$, $t \ge 0$.
(c) Let $F(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2$ be a primitive of $f(s)$ and define the functional

$$J(v) = \tfrac{1}{2}\|\nabla v\|^2 + \int_\Omega F(v)\,\mathrm{d}x, \quad v \in H^1.$$

Show that $J(u(\cdot, t)) \le J(u_0)$, $t \ge 0$.

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1. (a) Eigenfunction expansion:

$$u(t) = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} \hat{v}_j \varphi_j.$$

Parseval:

$$\|\nabla u(t)\|^2 = \sum_{j=1}^{\infty} \lambda_j \mathrm{e}^{-2\lambda_j t} \hat{v}_j^2 = \tfrac{1}{2} t^{-1} \sum_{j=1}^{\infty} (2\lambda_j t) \mathrm{e}^{-2\lambda_j t} \hat{v}_j^2 \le \tfrac{1}{2\mathrm{e}} t^{-1} \sum_{j=1}^{\infty} \hat{v}_j^2 = \tfrac{1}{2\mathrm{e}} t^{-1} \|v\|^2,$$

because $\max_{x \ge 0} x\mathrm{e}^{-x} = \mathrm{e}^{-1}$ is attained for $x = 1$. Parseval again:

$$\begin{aligned}
\int_0^t \|\nabla u(s)\|^2 \,\mathrm{d}s &= \int_0^t \sum_{j=1}^{\infty} \lambda_j \mathrm{e}^{-2\lambda_j s} \hat{v}_j^2 \,\mathrm{d}s = \sum_{j=1}^{\infty} \int_0^t \lambda_j \mathrm{e}^{-2\lambda_j s} \,\mathrm{d}s \, \hat{v}_j^2 \\
&= \sum_{j=1}^{\infty} \Big[-\tfrac{1}{2} \mathrm{e}^{-2\lambda_j s} \Big]_{s=0}^{t} \hat{v}_j^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} \Big(1 - \mathrm{e}^{-2\lambda_j t} \Big) \hat{v}_j^2 \\
&\le \tfrac{1}{2} \sum_{j=1}^{\infty} \hat{v}_j^2 = \tfrac{1}{2} \|v\|^2.
\end{aligned}$$

(b) Energy method, weak formulation:

$$u(t) \in H_0^1; \quad (u_t, \phi) + (\nabla u, \nabla \phi) = 0 \quad \forall \phi \in H_0^1, \ t > 0.$$

Take $\phi = u(t) \in H_0^1$:

$$\begin{aligned}
&(u_t, u) + (\nabla u, \nabla u) = 0, \\
&\tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2 + \|\nabla u\|^2 = 0, \\
&\tfrac{1}{2} \|u(t)\|^2 + \int_0^t \|\nabla u\|^2 \,\mathrm{d}s = \tfrac{1}{2} \|v\|^2, \\
&\int_0^t \|\nabla u\|^2 \,\mathrm{d}s \le \tfrac{1}{2} \|v\|^2.
\end{aligned}$$

Take $\phi = t u_t(t) \in H_0^1$:

$$\begin{aligned}
&t(u_t, u_t) + t(\nabla u, \nabla u_t) = 0, \\
&t\|u_t\|^2 + \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big(t \|\nabla u\|^2 \big) - \tfrac{1}{2} \|\nabla u\|^2 = 0, \\
&\int_0^t s\|u_t\|^2 \,\mathrm{d}s + \tfrac{1}{2} t \|\nabla u(t)\|^2 = \tfrac{1}{2} \int_0^t \|\nabla u\|^2 \,\mathrm{d}s,
\end{aligned}$$

so that, by the first part,

$$\begin{aligned}
&t\|\nabla u(t)\|^2 \le \int_0^t \|\nabla u\|^2 \,\mathrm{d}s \le \tfrac{1}{2} \|v\|^2, \\
&\|\nabla u(t)\|^2 \le \tfrac{1}{2} t^{-1} \|v\|^2.
\end{aligned}$$

2. See Chapter 8.

3. Assume that $v \in \mathcal{C}^1(\bar{\Omega})$. Let $\hat{r}(\theta) = (\cos\theta, \sin\theta)$ be the unit vector in the radial direction. Then

$$\begin{aligned} R^2\, v(R\cos\theta, R\sin\theta)^2 &= \int_0^R \frac{\partial}{\partial r}\Big(r\, v(r\cos\theta, r\sin\theta)\Big)^2 \,\mathrm{d}r \\ &= 2\int_0^R \Big(r\, v(r\cos\theta, r\sin\theta)\Big)\Big(v(r\cos\theta, r\sin\theta) + r\,\nabla v(r\cos\theta, r\sin\theta)\cdot\hat{r}(\theta)\Big)\,\mathrm{d}r \\ &\le 2\int_0^R \Big(v(r\cos\theta, r\sin\theta)^2 + R\,|v(r\cos\theta, r\sin\theta)|\,|\nabla v(r\cos\theta, r\sin\theta)|\Big) r\,\mathrm{d}r. \end{aligned}$$

Recall that $\mathrm{d}x = r\,\mathrm{d}r\,\mathrm{d}\theta$ and $\mathrm{d}s = R\,\mathrm{d}\theta$. Integrate with respect to θ:

$$\begin{aligned} R&\int_0^{2\pi} v(R\cos\theta, R\sin\theta)^2 R\,\mathrm{d}\theta \\ &\le 2\int_0^{2\pi}\int_0^R \Big(v(r\cos\theta, r\sin\theta)^2 + R\,|v(r\cos\theta, r\sin\theta)|\,|\nabla v(r\cos\theta, r\sin\theta)|\Big) r\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned}$$

This means

$$R\int_\Gamma v^2\,\mathrm{d}s \le 2\int_\Omega (v^2 + R|v|\,|\nabla v|)\,\mathrm{d}x.$$

With the Cauchy-Schwarz and Young's inequalities:

$$\begin{aligned} R\|v\|^2_{L_2(\Gamma)} &\le 2\int_\Omega (v^2 + R|v|\,|\nabla v|)\,\mathrm{d}x \\ &\le 2\|v\|^2_{L_2(\Omega)} + 2R\|v\|_{L_2(\Omega)}\|\nabla v\|_{L_2(\Omega)} \\ &\le 2\|v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Omega)} + R^2\|\nabla v\|^2_{L_2(\Omega)}. \end{aligned}$$

Therefore

$$\|v\|^2_{L_2(\Gamma)} \le 3R^{-1}\|v\|^2_{L_2(\Omega)} + R\|\nabla v\|^2_{L_2(\Omega)} \quad \forall v \in \mathcal{C}^1(\bar{\Omega}).$$

4. See Chapter 5.

5. (a) Let $V = H^1$. Find $u(t) \in V$ and $w(t) \in V$ such that

$$\begin{aligned} &(u_t, \phi) + (\nabla w, \nabla\phi) = 0 && \forall \phi \in V,\ t > 0, \\ &(w, \psi) = (\nabla u, \nabla\psi) + (f(u), \psi) && \forall \psi \in V,\ t > 0, \\ &u(0) = u_0. \end{aligned}$$

(b) Take $\phi = 1 \in V$ in the first equation and note that $\nabla\phi = \nabla 1 = 0$:

$$(u_t, 1) = 0,$$

that is,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(\cdot,t)\,\mathrm{d}x &= (u_t, 1) = 0, \\ \int_\Omega u(\cdot,t)\,\mathrm{d}x &= \int_\Omega u_0\,\mathrm{d}x. \end{aligned}$$

(c) Take $\phi = w(t) \in V$ and $\psi = u_t(t) \in V$:

$$\begin{aligned} &(u_t, w) + \|\nabla w\|^2 = 0, \\ &(w, u_t) = (\nabla u, \nabla u_t) + (f(u), u_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tfrac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,\mathrm{d}x\Big) = \frac{\mathrm{d}}{\mathrm{d}t}J(u). \end{aligned}$$

By combining these we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} J(u) + \|\nabla w\|^2 = 0, \\
&J(u(t)) - J(u(0)) + \int_0^t \|\nabla w\|^2 \,\mathrm{d}s = 0, \\
&J(u(t)) \le J(u_0).
\end{aligned}
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