Matematik Chalmers

Tentamen i
TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2010–08–27 fm V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.

Grades: 3: 20–29p, 4: 30–39p, 5: 40–. The exams are returned on Thursday September 9, 12–13, in Stig Larsson's office.

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1. The Klein-Gordon equation $u_{tt} - \Delta u + u = 0$ is studied in quantum field theory. Consider the initial-boundary value problem:

$$\begin{aligned} &u_{tt} - \Delta u + u = 0, && \text{in } \Omega \times \mathbf{R}_+, \\ &u = 0, && \text{on } \Gamma \times \mathbf{R}_+, \\ &u(\cdot, 0) = v,\ u_t(\cdot, 0) = w, && \text{in } \Omega. \end{aligned}$$

(a) Give a weak formulation of this problem.

(b) Define a suitable energy for this problem and show that the energy is conserved.

2. Consider the heat equation with Neumann's boundary condition,

$$\begin{aligned} &u_t - \Delta u = 0 && \text{in } \Omega \times \mathbf{R}_+, \\ &\partial u/\partial n = 0 && \text{on } \Gamma \times \mathbf{R}_+, \\ &u(\cdot, 0) = v && \text{in } \Omega. \end{aligned}$$

(a) Give a weak formulation of this problem.

(b) Show that $\overline{u(t)} = \overline{v}$ for $t \geq 0$, where $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x$ denotes the average of a function f.

(c) Show that there is a constant $\alpha > 0$ such that

$$\|u(t) - \overline{v}\| \leq \mathrm{e}^{-\alpha t}\|v - \overline{v}\|.$$

Hint: consider $w = u - \overline{u}$ and recall the inequality (from Problem 3.5 in the book)

$$\|v\|^2 \leq C\Big\{\|\nabla v\|^2 + \Big(\int_\Omega v\,\mathrm{d}x\Big)^2\Big\} \quad \forall v \in H^1.$$

3. Write the initial value problem for the wave equation

$$\begin{aligned} &u_{tt} - u_{xx} = 0 && \text{in } \mathbf{R} \times \mathbf{R}_+, \\ &u(\cdot, 0) = u_0,\ u_t(\cdot, 0) = u_1 && \text{in } \mathbf{R}, \end{aligned}$$

as a strictly hyperbolic system and solve it by the method of characteristics.

4. (a) Formulate the finite element method for the elliptic problem:

$$\begin{aligned} -\nabla \cdot \big(a\nabla u\big) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

(b) Formulate sufficient assumptions and prove the error estimate

$$|u_h - u|_1 \leq Ch\|u\|_2.$$

5. For the problem in Question 4, state and prove an "a posteriori" error estimate of the form

$$\|u_h - u\| \leq C\Big(\sum_{K \in \mathcal{T}_h} R_K^2\Big)^{1/2}$$

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TMA026/MMA430 Partial differential equations II
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1. (a) Find $u(t) \in H_0^1$ such that $u(0) = v$, $u_t(0) = w$ and

$$(u_{tt}, \phi) + (\nabla u, \nabla \phi) + (u, \phi) = 0, \quad \forall \phi \in H_0^1,\ t > 0.$$

(b) Choose $\phi = u_t(t)$:

$$\begin{aligned}
&(u_{tt}, u_t) + (\nabla u, \nabla u_t) + (u, u_t) = 0, \\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_t\|^2 + \|\nabla u\|^2 + \|u\|^2\big) = 0, \\
&\frac{1}{2}\big(\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^2\big) = \frac{1}{2}\big(\|w\|^2 + \|\nabla v\|^2 + \|v\|^2\big).
\end{aligned}$$

This means that the energy $E(t) = \frac{1}{2}\big(\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|^2\big)$ is conserved.

2. (a) Find $u(t) \in H^1$ such that $u(0) = v$ and

$$(u_t, \phi) + (\nabla u, \nabla \phi) = 0, \quad \forall \phi \in H^1,\ t > 0.$$

(b) Take $\phi = 1 \in H^1$:

$$\begin{aligned}
&(u_t, 1) + (\nabla u, \nabla 1) = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}(u, 1) = 0 \\
&(u(t), 1) = (v, 1)
\end{aligned}$$

which is the desired result because $(u(t), 1) = \int_\Omega u(t)\,\mathrm{d}x = |\Omega|\,\overline{u(t)}$.
(c) The function $w = u - \overline{u} = u - \overline{v}$ satisfies $w_t - \Delta w = 0$, $w(0) = v - \overline{v}$. Therefore

$$(w_t, \phi) + (\nabla w, \nabla \phi) = 0, \quad \forall \phi \in H^1,\ t > 0.$$

Take $\phi = w(t)$ and note that from Problem 3.5 in the book $\|w(t)\|^2 \le C(\|\nabla w(t)\|^2 + \overline{w(t)}^2) = C\|\nabla w(t)\|^2$, because $\overline{w(t)} = 0$. Then, with $\alpha = C$,

$$\begin{aligned}
&(w_t, w) + \|\nabla w\|^2 = 0 \\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \alpha\|w\|^2 \le 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{2\alpha t}\|w(t)\|^2\Big) \le 0 \\
&\|w(t)\|^2 \le \mathrm{e}^{-2\alpha t}\|w(0)\|^2 \\
&\|u(t) - \overline{v}\| \le \mathrm{e}^{-\alpha t}\|v - \overline{v}\|
\end{aligned}$$

3. See Example 11.9.

4. See Chapter 5.

5. See Chapter 5.

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