Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2013–05–28 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Formulate the upwind scheme for the problem

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} \quad \text{in } \mathbf{R}\times\mathbf{R}_+; \quad u(\cdot,0)=v \quad \text{in } \mathbf{R}. \qquad (a>0)$$

Formulate and prove a stability result and an error estimate.

2. Maxwell's equations may be written (after elimination of the magnetic field H)

$$\frac{\partial^2 E}{\partial t^2} + \nabla\times(\nabla\times E) = f \quad \text{in } \Omega\times\mathbf{R}_+,$$

where $E=E(x,t)\in\mathbf{R}^3$ is the electric field, $\nabla\times E = \operatorname{rot} E$, and $f=f(x,t)\in\mathbf{R}^3$ is a source term. This should be complemented by initial and boundary conditions. I wrote this in weak form as

$$\begin{aligned}
&E(t)\in H^1(\Omega)^3,\ E(0)=E_0,\ E_t(0)=E_1,\\
&(E_{tt},v)+(\nabla\times E,\nabla\times v)=(f,v)+(g,v)_\Gamma \quad \forall v\in H^1(\Omega)^3,\ t>0.
\end{aligned}$$

Here $(f,v)=\int_\Omega f\cdot v\,\mathrm{d}x$ and $(g,v)_\Gamma=\int_\Gamma g\cdot v\,\mathrm{d}s$. Repeat my derivation and identify what boundary condition I used. Hint: recall the product rule $\nabla\cdot(u\times v)=(\nabla\times u)\cdot v - u\cdot(\nabla\times v)$ and use this together with the divergence theorem to obtain an integration by parts formula.

3. Let $u_h(t)=E_h(t)v_h$ denote the solution of the semi-discrete finite element equation corresponding to the homogeneous heat equation as in the book. Prove that

$$\int_0^t \|\nabla u_h(s)\|^2\,\mathrm{d}s \le C\|v_h\|^2, \qquad \|\nabla u_h(t)\|^2 \le Ct^{-1}\|v_h\|^2.$$

4. Consider the wave equation

$$\begin{aligned}
u_{tt}-\Delta u &= f \quad \text{in } \Omega\times\mathbf{R}_+,\\
u &= 0 \quad \text{on } \Gamma\times\mathbf{R}_+,\\
u(0)=v,\ u_t(0) &= w \quad \text{in } \Omega.
\end{aligned}$$

(a) Set $U=\begin{bmatrix}U_1\\U_2\end{bmatrix}=\begin{bmatrix}u\\u_t\end{bmatrix}$ and write the problem as a system of the form (two equations first order in time)

$$\begin{aligned}
U_t+\mathcal{A}u &= F \quad \text{in } \Omega\times\mathbf{R}_+,\\
U &= 0 \quad \text{on } \Gamma\times\mathbf{R}_+,\\
U(0) &= U_0 \quad \text{in } \Omega.
\end{aligned}$$

(Here $\mathcal{A}$ is a second order differential operator matrix with respect to x.)

(b) Write the system in weak form by multiplying by $\begin{bmatrix}-\Delta V_1\\V_2\end{bmatrix}$ and integrating by parts. Hint: a natural scalar product on $(H_0^1)^2$ is then $\langle U,V\rangle=(\nabla U_1,\nabla V_1)+(U_2,V_2)$. Use this to prove the usual energy identity when $f=0$. What do you get when $f\neq 0$?

Continued on page 2!

5. Consider the Robin problem

$$\begin{aligned} -\nabla \cdot (a\nabla u) &= f \quad \text{in } \Omega, \\ a\frac{\partial u}{\partial n} + h(u-g) &= k \quad \text{on } \Gamma. \end{aligned}$$

Formulate the piecewise linear finite element method. Prove the *a posteriori* error bound

$$\|u_h - u\| \le C\Big(\sum_{K\in\mathcal{T}_h} R_K^2\Big)^{1/2},$$

$$R_K = h_K^2\| - \nabla\cdot(a\nabla u_h) - f\|_K + h_K^{3/2}\|a[n\cdot\nabla u_h]\|_{\partial K\setminus\Gamma} + h_K^{3/2}\|an\cdot\nabla u_h - hg - k\|_{\partial K\cap\Gamma},$$

where $[n\cdot\nabla u_h]$ denotes the jump across ∂K in the normal derivative $n\cdot\nabla u_h$. Hint: use the adjoint problem $-\nabla\cdot(a\nabla\Phi) = e$ in Ω, $a\frac{\partial\Phi}{\partial n} + h\Phi = 0$ on Γ, which has the same regularity as the Dirichlet problem. (This is corrected. The original problem was incorrectly formulated.)

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1. See Chapter 12.

2. $$\int_\Omega \big((\nabla\times u)\cdot v - u\cdot(\nabla\times v)\big)\,dx = \int_\Omega \nabla\cdot(u\times v)\,dx$$

$$= \{\text{div thm}\} = \int_\Gamma n\cdot(u\times v)\,dS$$

leads to

$$\int_\Omega (\nabla\times u)\cdot v\,dx = \int_\Gamma n\cdot(u\times v)\,dS + \int_\Omega u\cdot(\nabla\times v)\,dx$$

Use this with $u = \nabla\times E$.

$$(f,v) = (E_{tt} + \nabla\times(\nabla\times E), v) =$$

$$= (E_{tt}, v) + \int_\Gamma \underbrace{n\cdot((\nabla\times E)\times v)}_{=-g\cdot v}\,dS + (\nabla\times E, \nabla\times v)$$

Here I used $-g\cdot v = n\cdot((\nabla\times E)\times v) =$

$$= v\cdot(n\times(\nabla\times E)),$$

that is, $g = -n\times(\nabla\times E) = (\nabla\times E)\times n$ on Γ

Hence
$$\begin{cases} E \in (H^1)^3 \\ (E_{tt}, v) + (\nabla\times E, \nabla\times v) = (f,v) + (g,v), \quad \forall v \in (H^1)^3 \end{cases}$$

$(\nabla\times E)\times n = g$ on Γ

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3. $\begin{cases} (u_{n,t}, \chi) + (\nabla u_n, \nabla \chi) = 0 \\ u_n(0) = v_n \end{cases}$

a) $\chi = u_n(t)$: $\quad \frac{1}{2} D_t \|u_n\|^2 + \|\nabla u_n\|^2 = 0$

$$\frac{1}{2}\|u_n(t)\|^2 + \int_0^t \|\nabla u_n\|^2 ds = \frac{1}{2}\|v_n\|^2$$

b) $\chi = t\, u_{n,t}(t)$: $\quad t\|u_{n,t}\|^2 + \underbrace{t(\nabla u_n, \nabla u_{n,t})}_{} = 0$

$$= \frac{1}{2} D_t\left(t\|\nabla u_n\|^2\right) - \frac{1}{2}\|\nabla u_n\|^2$$

$$t\|u_{n,t}\|^2 + \frac{1}{2} D_t\left(t\|\nabla u_n\|^2\right) = \frac{1}{2}\|\nabla u_n\|^2$$

$$\int_0^t s\|u_{n,t}\|^2 ds + \frac{1}{2} t\|\nabla u_n(t)\|^2 = \frac{1}{2}\int_0^t \|\nabla u_n\|^2 ds$$

$$\leq \frac{1}{4}\|v_n\|^2$$

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4. $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} u \\ u_t \end{bmatrix} \qquad U_t = \begin{bmatrix} u_t \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_t \\ \Delta u + f \end{bmatrix} = \begin{bmatrix} U_2 \\ \Delta U_1 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$

$$= -\begin{bmatrix} 0 & -I \\ -\Delta & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$\begin{cases} U_t + \mathcal{A} U = F \\ U(0) = U_0 \end{cases} \qquad \mathcal{A} = \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix}$

$$U_0 = \begin{bmatrix} v \\ w \end{bmatrix}$$

Weak form: mult by $\begin{bmatrix} -\Delta V_1 \\ V_2 \end{bmatrix}$:

$$\begin{cases} U(t) \in (H_0^1)^2 \\ b(U_t, V) + a(U, V) = L(V) \quad \forall V \in (H_0^1)^2 \end{cases}$$

where $b(U_t, V) = (\nabla U_{1,t}, \nabla V_1) + (U_{2,t}, V_2)$

$$a(U, V) = -(\nabla U_2, \nabla V_1) + (\nabla U_1, \nabla V_2)$$

$$L(V) = (f, V_2)$$

With $f = 0$, $V = U$:

$$b(U_t, U) + \underbrace{a(U, U)}_{=0} = 0$$

$$\frac{1}{2} D_t\, b(U, U) = 0$$

$$\frac{1}{2} b(U, U) = \frac{1}{2} b(U_0, U_0)$$

$$\frac{1}{2}\left(\|\nabla U_1\|^2 + \|U_2\|^2\right) = \text{const.}$$

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5. Similar to Theorem 5.6.

See the next page.

5. $\quad \begin{cases} u \in H^1 \\ a(u,v) = (f,v) + (k+hg, v)_\Gamma \quad \forall v \in H^1 \end{cases}$ $\quad$ 4

$$\begin{cases} \text{FEM: } u_h \in \tilde{S}_h \subset H^1 \\ a(u_h, \chi) := (a\nabla u_h, \nabla\chi) + (h u_h, \chi)_\Gamma = (f,\chi) + (k+hg, \chi)_\Gamma \quad \forall \chi \in \tilde{S}_h \end{cases}$$

$$\begin{cases} \text{Adjoint problem: } \phi \in H^1 \\ a(v,\phi) := (a\nabla v, \nabla\phi) + (h v, \phi)_\Gamma = (v, e) \quad \forall v \in H^1 \end{cases}$$

$v = e = u_h - u$:

$$\|e\|^2 = a(e,\phi) = (a\nabla(u_h - u), \nabla\phi) + (h(u_h - u), \phi)_\Gamma$$

$$= \sum_K (a\nabla u_h, \nabla\phi)_K + (h u_h, \phi)_\Gamma \underbrace{- (a\nabla u, \nabla\phi) - (h u, \phi)_\Gamma}_{= -(f,\phi) - (k+hg, \phi)_\Gamma}$$

$$= \sum_K (A u_h - f, \phi)_K - \frac{1}{2}\left(a\left[\frac{\partial u_h}{\partial n}\right], \phi\right)_{\partial K \setminus \Gamma} + \left(a\frac{\partial u_h}{\partial n}, \phi\right)_{\partial K \cap \Gamma} - (k+hg, \phi)_\Gamma$$

$$= \sum_K \left\{ (A u_h - f, \phi)_K - \frac{1}{2}\left(a\left[\frac{\partial u_h}{\partial n}\right], \phi\right)_{\partial K \setminus \Gamma} + \left(a\frac{\partial u_h}{\partial n} - k - hg, \phi\right)_{\partial K \cap \Gamma} \right\}$$

Replace ϕ by $\phi - I_h\phi$:

$$\|e\|^2 \le \sum_K \|A u_h - f\|_K \|\phi - I_h\phi\|_K + \frac{1}{2}\left\|a\left[\frac{\partial u_h}{\partial n}\right]\right\|_{\partial K \setminus \Gamma} \|\phi - I_h\phi\|_{\partial K \setminus \Gamma} + \left\|a\frac{\partial u_h}{\partial n} - hg - k\right\|_{\partial K \cap \Gamma} \|\phi - I_h\phi\|_{\partial K \cap \Gamma}$$

$$\le \sum_K \left(h_K^2 \|A u_h - f\|_K + \frac{1}{2} h_K^{3/2} \left\|a\left[\frac{\partial u_h}{\partial n}\right]\right\|_{\partial K \setminus \Gamma} + h_K^{3/2} \left\|a\frac{\partial u_h}{\partial n} - hg - k\right\|_{\partial K \cap \Gamma} \right) |\phi|_{2,K}$$

$$\le \left(\sum_K R_K^2\right)^{1/2} \|\phi\|_2 \le C \left(\sum R_K^2\right)^{1/2} \|e\|.$$

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