Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2013–08–30 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Formulate the trace theorem and prove it for a square domain in $\mathbf{R}^2$.

2. The damped wave equation is

$$\begin{aligned}&u_{tt} - \Delta u + \alpha u_t = f \quad \text{in } \Omega \times \mathbf{R}_+\\ &u = 0 \quad \text{on } \Gamma \times \mathbf{R}_+\\ &u(\cdot,0) = v,\ u_t(\cdot,0) = w \quad \text{in } \Omega\end{aligned}$$

Here α is a positive constant. (a) Explain why the extra term αu_t represents "damping".
(b) Prove a bound for the energy $\mathcal{E}(t) = \frac{1}{2}(\|u_t(t)\|^2 + \|\nabla u(t)\|^2)$.

3. Consider the spatially semidiscrete finite element approximation of the heat equation

$$\begin{aligned}&u_t - \Delta u = f, && \text{in } \Omega \times \mathbf{R}_+,\\ &u = 0, && \text{on } \Gamma \times \mathbf{R}_+,\\ &u(\cdot,0) = v, && \text{in } \Omega.\end{aligned}$$

Show the error estimate

$$\|u_h(t) - u(t)\| \le \|v_h - v\| + Ch^2\Big\{\|v\|_2 + \int_0^t \|u_t(s)\|_2 \,\mathrm{d}s\Big\}.$$

4. The Stokes equations in $\Omega \subset \mathbf{R}^3$ are

$$\begin{aligned}-\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega\\ \mathbf{u} &= 0 \quad \text{on } \Gamma\end{aligned} \tag{1}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the pressure, and $\mathbf{f} = (f_1, f_2, f_3)$ is a force.
(a) Show that the Stokes equations can be given the following weak formulation. Find $\mathbf{u} \in X$ and $p \in M$ such that

$$\begin{aligned}&a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in X\\ &b(\mathbf{u}, q) = 0 \quad \forall q \in M\end{aligned} \tag{2}$$

Here $X = (H_0^1)^3$, $M = \{q \in L_2 : \bar{q} = \frac{1}{|\Omega|}\int_\Omega q \,\mathrm{d}x = 0\}$ and

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega \sum_{i,j} \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,\mathrm{d}x, \quad b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q), \quad (\mathbf{f}, \mathbf{v}) = \int_\Omega \sum_j f_j v_j \,\mathrm{d}x$$

(b) Show that if a weak solution is extra smooth, then it is a classical solution, that is, a solution of (1). Hint: to show $\nabla \cdot \mathbf{u} = 0$, take $q = \nabla \cdot \mathbf{u} - \overline{\nabla \cdot \mathbf{u}}$ in (2).
(c) Use the weak formulation to prove a bound for the velocity.

5. State and prove the maximum principle for the heat equation.

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1. See the book.

2. (a) If we move the term to the right side, we get $f - \alpha u_t$. We see that we have an additional force which is proportional to the velocity with negative coefficient.

In (b) we also see that $D_t \mathcal{E}(t) = -\alpha \|u_t\|^2 < 0$ when $f = 0$

(b) Multiply by u_t:

$$(u_{tt}, u_t) - (\Delta u, u_t) + \alpha (u_t, u_t) = (f, u_t)$$

$$(u_{tt}, u_t) + (\nabla u, \nabla u_t) + \alpha \|u_t\|^2 = (f, u_t)$$

$$D_t\left(\tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\|\nabla u\|^2\right) + \alpha \|u_t\|^2 = (f, u_t) \le \|f\|\,\|u_t\|$$

$$\le \frac{1}{2\alpha}\|f\|^2 + \frac{\alpha}{2}\|u_t\|^2$$

$$D_t \mathcal{E}(t) + \underbrace{\frac{\alpha}{2}\|u_t\|^2}_{\ge 0} \le \frac{1}{2\alpha}\|f\|^2$$

$$\mathcal{E}(t) \le \mathcal{E}(0) + \frac{1}{2\alpha}\int_0^t \|f(s)\|^2\, ds$$

3. See the book.

4. (a) $(f, v) = \int_\Omega \sum f_i v_i \, dx = \sum \int_\Omega \left(-\Delta u_i v_i + \frac{\partial p}{\partial x_i} v_i\right) dx =$

$v = 0$ on Γ

$$= \sum_i \int_\Omega \left(\nabla u_i \cdot \nabla v_i - p \frac{\partial v_i}{\partial x_i}\right) dx =$$

$$= \int_\Omega \sum_i \sum_j \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx - (p, \nabla \cdot v)$$

$$= a(u, v) + b(v, p) \quad \forall v \in X$$

$$0 = (\nabla \cdot u, q) = -b(u, q) \quad \forall q \in M$$

(b) Take $q = \nabla \cdot u - \overline{\nabla \cdot u}$. Then $\bar{q} = 0$, so

(2) gives $0 = b(u, q) = b(u, \nabla \cdot u) - b(u, \overline{\nabla \cdot u}) =$

$$= -\int_\Omega |\nabla \cdot u|^2 dx + \overline{\nabla \cdot u} \underbrace{\int_\Omega \nabla \cdot u \, dx}_{=0 \text{ because } u = \text{ on } \Gamma} = -\|\nabla \cdot u\|^2$$

Therefore: $\|\nabla\cdot u\|^2 = 0$ and $\nabla\cdot u = 0$.

If we do the integration by parts from (a) backwards then

$$a(u,v) + b(v,p) = (f,v) \qquad \forall v \in X$$

$$\int_\Omega (-\Delta u + \nabla p - f)\cdot v \, dx = 0 \qquad \forall v \in X$$

$$-\Delta u + \nabla p - f = 0 \quad \text{in } \Omega$$

(c) Take $v = u$, $q = p$ in (2):

$$\begin{cases} a(u,u) + b(u,p) = (f,u) \\ b(u,p) = 0 \end{cases}$$

$$\Rightarrow \quad a(u,u) = (f,u)$$

$$\Rightarrow \quad |u|_1^2 = (f,u) \le \|f\|\,\|u\| \overset{\text{Poincaré}}{\le} C\|f\|\,|u|_1$$

$$\Rightarrow \quad |u|_1 \le C\|f\|$$

Here $|u|_1^2 = \int_\Omega \sum_{ij} \left|\frac{\partial u_i}{\partial x_j}\right|^2 dx = \sum_i \|\nabla u_i\|^2 = \sum_i |u_i|_1^2$

so Poincaré can be used.

5. See the book.

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