Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2014–05–27 f V

SOLUTIONS

1.

(a) Find $u = u_0 + g$ where $u_0 \in H_0^1(\Omega)$ solves

$$(\nabla u_0, \nabla v) = -(\nabla g, \nabla v), \quad \forall v \in H_0^1(\Omega).$$

(b) Since $-\Delta u = 0$, u attains both max and min on the boundray by the maximum principle. Therefore $\|u\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}$.

(c) Let $V_h = \{v \in C(\Omega) :$ piecewise linear on a mesh $\mathcal{T}_h\}$. Find $U = I_h g + U_0$ where $U_0 \in V_h$ solves $(\nabla U_0, \nabla v) = -(\nabla I_h g, \nabla v), \forall v \in V_h$.

It holds $(\nabla(u_0 - U_0), \nabla v) = (\nabla(I_h g - g), \nabla v)$ for all $v \in V_h$. We conclude,

$$\begin{aligned}\|\nabla(u_0 - U_0)\|_{L^2(\Omega)}^2 &\le (\nabla(u_0 - U_0), \nabla(u_0 - I_h u_0)) + (\nabla(I_h g - g), \nabla(U_0 - I_h u_0)) \\ &\le Ch\|\nabla(u_0 - U_0)\|_{L^2(\Omega)}\|D^2 u_0\|_{L^2(\Omega)} \\ &\quad + \|\nabla(I_h g - g)\|_{L^2(\Omega)}\left(\|\nabla(u_0 - I_h u_0)\|_{L^2(\Omega)} + \|\nabla(u_0 - U_0)\|_{L^2(\Omega)}\right)\end{aligned}$$

We get $\|\nabla(u_0 - U_0)\|_{L^2(\Omega)} \le Ch\|D^2 u_0\|_{L^2(\Omega)} + C\|\nabla(I_h g - g)\|_{L^2(\Omega)}$, and therefore,

$$\|\nabla(u - U)\|_{L^2(\Omega)} \le Ch\left(\|D^2 u\|_{L^2(\Omega)} + \|D^2 g\|_{L^2(\Omega)}\right) + C\|\nabla(I_h g - g)\|_{L^2(\Omega)},$$

2.

(a) Let $\{\phi_i\}_{i=1}^\infty$ denote the eigenfunctions to $-\Delta$, with corresponding eigenvalues $\{\lambda_i\}_{i=1}^\infty$. Then $u(t) = \sum_{i=1}^\infty e^{-\lambda_i t}(v, \phi_i)\phi_i$.

(b) By Parseval's equality

$$\|u(t)\|_{L^2(\Omega)} = \left(\sum_{i=1}^\infty e^{-2\lambda_i t}(v, \phi_i)^2\right)^{1/2} \le e^{-\lambda_1 t}\|v\|_{L^2(\Omega)}.$$

(c) We have that $|u|_{H^1(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i e^{-2\lambda_i t}(v, \phi_i)^2 \le \sum_{i=1}^\infty \lambda_i (v, \phi_i)^2 = |v|_{H^1(\Omega)}^2$. Furthermore, $|u|_{H^1(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i e^{-2\lambda_i t}(v, \phi_i)^2 \le t^{-1}\sum_{i=1}^\infty \lambda_i t e^{-2\lambda_i t}(v, \phi_i)^2 \le Ct^{-1}\|v\|_{L^2(\Omega)}^2$.

3.

(a) We have $\|f(u) - f(w)\|_{L^2(\Omega)} \le \|u - w\|_{L^2(\Omega)} + \|u + v\|_{L^4(\Omega)}\|u - w\|_{L^4(\Omega)} \le (1 + C2R)\|u - w\|_{H^1(\Omega)}$.

(b) We have

$$\begin{aligned}\|Su(t)\|_{H^1(\Omega)} &\le \|v\|_{H^1(\Omega)} + \int_0^t (t-s)^{-1/2}\|f(u(s))\|_{L^2(\Omega)}\,ds \\ &\le R_0 + C\tau^{1/2}R,\end{aligned}$$

since $\|f(u)\|_{L^2(\Omega)} \le \|f(u) - f(0)\|_{L^2(\Omega)} \le C\|u\|_{H^1(\Omega)} \le C \cdot R$. Let $R = 2R_0$ and τ be small enough for $C\tau^{1/2}R \le R_0$.

(c) We have,

$$\begin{aligned}\|Su - Sw\|_{H^1(\Omega)} &\le \int_0^t (t-s)^{-1/2}\|f(u(s)) - f(w(s))\|_{L^2(\Omega)}\,ds \\ &\le C(R)\tau^{1/2}\max_{0\le t\le\tau}\|u(t) - w(t)\|_{H^1(\Omega)}.\end{aligned}$$

By taking $\max_{0\leq t\leq \tau}$ and choosing τ small enough for $\tau^{1/2}C(R) < 1$ we prove that S is a contraction.

4.

(a) Let $\{v_i\}_{i=1}^{\infty} \in H_0^1(\Omega)$ be a sequence with limit $v \notin H_0^1(\Omega)$ i.e. $\|\gamma v\|_{L^2(\Gamma)} = \delta > 0$. For any $\epsilon > 0$ there exists an n such that,

$$C\|v_i - v\|_{H^1(\Omega)} \leq \epsilon.$$

Using the trace theorem we get,

$$\delta = \|\gamma v\|_{L^2(\Gamma)} = \|\gamma(v - v_i)\|_{L^2(\Gamma)} \leq C\|v_i - v\|_{H^1(\Omega)} \leq \epsilon,$$

for all $i > n$. By choosing $\epsilon < \delta$ we get a contradiction i.e. $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$ and therefore a Hilbert space.

(b) a should be coercive and bounded and l should be bounded.

(c) Since $(b \cdot \nabla u, u) = -((\nabla \cdot b)u, u) - (u, b \cdot \nabla u)$ i.e.

$$(b \cdot \nabla u, u) = -\frac{1}{2}((\nabla \cdot b)u, u).$$

We pick a vector with no divergence, e.g. $\begin{bmatrix} y \\ x \end{bmatrix}$. We get $a(u, u) = (a\nabla u, \nabla u)$ and coercivity follows immedeatelly for any $0 < \alpha \leq a \leq \beta$. The boundedness follows since $\|b\|_{L^\infty(\Omega)} < \gamma$ in a bounded domain, $a(v, w) \leq \beta|u|_{H^1(\Omega)}|v|_{H^1(\Omega)} + \gamma|u|_{H^1(\Omega)}\|v\|_{L^2(\Omega)} \leq (\beta+\gamma)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$.

5. See proof of Theorem 10.1 in *Partial differential equations with numerical methods* by Thomée and Larsson.

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