Matematik Chalmers

TMA026 (MAN665) Partiella differentialekvationer fk, 2003–10–22 fm V

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Inga hjälpmedel. Kalkylator ej tillåten. Uppgifterna är värda 10 poäng vardera.

Ω is a bounded convex domain in $\mathbf{R}^2$ whose boundary Γ is a polygon and $\|v\| = \|v\|_{L_2(\Omega}$, $|v|_1 = \|\nabla v\|$, $\|v\|_k = \|v\|_{H^k(\Omega)}$.

1. (a) Formulate one of the maximum principles that we studied in the course.

(b) Present the main idea of its proof.

(c) Present one application of this maximum principle.

2. Let u be a solution of the initial-boundary value problem

$$\begin{aligned} u_t - \nabla\cdot(a(x)\nabla u) &= 0, && \text{in } \Omega\times\mathbf{R}_+,\\ u &= 0, && \text{in } \Gamma\times\mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

Make the usual assumptions about the coefficient a and show that

$$\begin{aligned} &\|u(t)\| \le \|v\|, && t\ge 0,\\ &\|u(t)\|_1 \le C\|v\|_1, && t\ge 0,\\ &\|u(t)\|_2 \le C\|v\|_2, && t\ge 0,\\ &|u(t)|_1 \le Ct^{-1/2}\|v\|, && t>0. \end{aligned}$$

3. Formulate the semidiscrete finite element method for the problem in Problem 2. Prove an error estimate.

4. Consider the initial-boundary value problem

$$\begin{aligned} u_t + a\cdot\nabla u + a_0 u &= f, && \text{in } \Omega\times\mathbf{R}_+,\\ u &= g, && \text{in } \Gamma_-\times\mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

(a) Present the method of characteristics for this problem.

(b) Prove an energy estimate for u under suitable assumptions on the coefficients a and a_0.

5. Consider the Navier-Stokes equations for the motion of an incompressible fluid: find a vector field $u = (u_1, u_2)$ and a scalar field p such that

$$\begin{aligned} u_t - \Delta u + (u\cdot\nabla)u + \nabla p &= f, && \text{in } \Omega\times\mathbf{R}_+,\\ \nabla\cdot u &= 0, && \text{in } \Omega\times\mathbf{R}_+,\\ u &= 0, && \text{on } \Gamma\times\mathbf{R}_+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned} \tag{1}$$

Here $f = f(x,t)$ and $v = v(x)$ are given vector fields and we define $u\cdot\nabla = \sum_{i=1}^{2} u_i \frac{\partial}{\partial x_i}$ so that $((u\cdot\nabla)u)_j = \sum_{i=1}^{2} u_i \frac{\partial u_j}{\partial x_i}$. Let $(L_2)^2 = \{v = (v_1, v_2) : v_i \in L_2\}$ with scalar product $(u,v) = \int_\Omega u\cdot v\,dx$ and $(H_0^1)^2 = \{v = (v_1,v_2) : v_i \in H_0^1\}$ with norm $|v|_1^2 = \sum_{i=1}^2\sum_{j=1}^2 \|\partial v_i/\partial x_j\|^2$.

(a) Show that, for all $p \in H^1$, $u, v, w \in (H_0^1)^2$,

$$\begin{aligned} (\nabla p, u) &= -(p, \nabla\cdot u),\\ ((u\cdot\nabla)v, w) &= -((\nabla\cdot u)v, w) - (v, (u\cdot\nabla)w). \end{aligned}$$

(b) Assume that u, p satisfy (1). Show that $((u\cdot\nabla)u, u) = 0$ and $(\nabla p, u) = 0$. Prove an energy estimate for u.

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TMA026 Partiella differentialekvationer fk, 2003–10–22. Lösningar.

Ω is a bounded convex domain in $\mathbf{R}^2$ whose boundary Γ is a polygon and $\|v\| = \|v\|_{L_2(\Omega}$, $|v|_1 = \|\nabla v\|$, $\|v\|_k = \|v\|_{H^k(\Omega)}$.

1. See the book.

2. The weak formulation is

$$\begin{aligned} &u(t) \in H_0^1(\Omega),\ u(0) = v, \\ &(u_t, \varphi) + a(u, \varphi) = 0, \quad \forall \varphi \in H_0^1(\Omega), \end{aligned} \tag{1}$$

where $a(\cdot,\cdot) = (a\nabla\cdot, \nabla\cdot)$. As usual we assume: $0 < a_0 \le a(x) \le a_1$ for all $x \in \bar{\Omega}$, so that

$$\begin{aligned} a(v,v) &\ge a_0 |v|_1^2, \quad \forall v \in H_0^1, \\ |a(v,w)| &\le a_1 |v|_1\, |w|_1, \quad \forall v, w \in H_0^1. \end{aligned}$$

Here $|v|_1 = \|\nabla v\|$.
(a) Take $\varphi = u$:

$$(u_t, u) + a(u,u) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + a(u,u) = 0,$$

$$\|u(t)\|^2 + 2\int_0^t a(u,u)\,ds = \|v\|^2, \tag{2}$$

$$\|u(t)\| \le \|v\|. \tag{3}$$

(b) Take $\varphi = u_t$:

$$\|u_t\|^2 + a(u, u_t) = 0,$$

$$\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}a(u,u) = 0, \qquad \text{[because } a(\cdot,\cdot) \text{ is symmetric]} \tag{4}$$

$$2\int_0^t \|u_t\|^2\,ds + a(u(t),u(t)) = a(v,v),$$

$$a(u(t),u(t)) \le a(v,v),$$

$$\begin{aligned} a_0\|\nabla u(t)\|^2 &\le a(u(t),u(t)) \le a(v,v) \\ &\le a_1\|\nabla v\|^2 + \|v\|^2 \le (a_1 + c)\|\nabla v\|^2, \end{aligned} \tag{5}$$

$$\|\nabla u(t)\| \le C\|\nabla v\|.$$

Together with (3) this proves

$$\|u(t)\|_1 \le \|v\|_1.$$

(c) Differentiate (1) with respect to t and then take $\varphi = u_t$:

$$(u_{tt}, u_t) + a(u_t, u_t) = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + a(u_t,u_t) = 0,$$

$$\|u_t(t)\|^2 + 2\int_0^t a(u_t,u_t)\,ds = \|u_t(0)\|^2,$$

$$\|u_t(t)\| \le \|u_t(0)\|.$$

Now $u_t = \nabla\cdot(a\nabla u) = a\Delta u + \nabla a \cdot \nabla u$. Therefore $\|u_t(0)\| = \|a\Delta v + \nabla a\cdot\nabla v\| \le C\|v\|_2$, so that

$$\|u_t(t)\| \le C\|v\|_2.$$

Also, using elliptic regularity and the previous estimates of $\|u_t(t)\|$ and $\|u(t)\|_1$,

$$\|u(t)\|_2 \le C\|\Delta u\| \le C\|a\Delta u\| = C\|u_t - \nabla a\cdot\nabla u\| \le C(\|u_t(t)\| + \|u(t)\|_1) \le C\|v\|_2.$$

(d) Multliply (4) by t:

$$
\begin{aligned}
&t\|u_t\|^2 + \frac{1}{2}t\frac{d}{dt}a(u,u) = 0,\\
&2t\|u_t\|^2 + \frac{d}{dt}\big(ta(u,u)\big) = a(u,u),\\
&2\int_0^t s\|u_t\|^2\,ds + ta(u(t),u(t)) = \int_0^t a(u,u)\,ds,\\
&a_0 t\|\nabla u(t)\|^2 \le ta(u(t),u(t)) \le \int_0^t a(u,u)\,ds \le \frac{1}{2}\|v\|^2, \quad \text{[by (2)]}\\
&\|\nabla u(t)\| \le Ct^{-1/2}\|v\|.
\end{aligned}
$$

3. See the book, essentially the same as in Chapter 10.

4. (a) The characteristics $x = x(s)$, $t = t(s)$ are given by

$$
\begin{aligned}
\frac{dx}{ds} &= a(x(s), t(s)),\\
\frac{dt}{ds} &= t.
\end{aligned}
$$

Hence $t = s + C$, choose $C = 0$ so that $s = 0$ at the initial time. Then $t = s$ and the first equation becomes

$$\frac{dx}{dt} = a(x(t), t).$$

Then $w(t) = u(x(t), t)$ satisfies the equation

$$\frac{dw(t)}{dt} + a_0(x(t), t)w(t) = f(x(t), t). \tag{6}$$

To find the solution at $(\bar{x}, \bar{t})$ we follow the characteristic thru $(\bar{x}, \bar{t})$ backwards until we hit $t = 0$ at x_0 or hit Γ_- at (x_0, t_0). Then we solve (6) with the initial condition $w(0) = v(x_0)$ or $w(t_0) = g(x_0)$.

(b) We assume $a_0(x) - \frac{1}{2}\nabla\cdot a(x) \ge \alpha > 0$. Then it is easy to show

$$\|u(t)\|^2 + \int_0^t\int_{\Gamma_+} u^2\, n\cdot a\,ds\,dt + \alpha\int_0^t \|u\|^2\,dt \le \|v\|^2 + \int_0^t \|f\|^2\,dt + \int_0^t\int_{\Gamma_-} g^2|n\cdot a|\,ds\,dt.$$

5. (a) Integrate by parts using Green's formula and use the fact that u, v, w are $= 0$ on Γ.

(b) That $((u\cdot\nabla)u, u) = 0$ and $(\nabla p, u) = 0$ follows directly from (a). Then the standard energy argument gives

$$\|u(t)\|^2 + \int_0^t |u|_1^2\,ds = \|v\|^2 + C\int_0^t \|f\|^2\,dt.$$

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