Matematik Chalmers

Tentamen i TMA026 Partiella differentialekvationer, fk, TM, 2005–10–19 fm M

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You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. State the trace theorem and prove it for a square domain in $\mathbf{R}^2$.

2. Give a weak formulation and prove existence and uniqueness of solutions to the boundary value problem

$$\begin{aligned} &-\nabla\cdot(a\nabla u) = f, && \text{in } \Omega,\\ &u = g, && \text{on } \Gamma, \end{aligned}$$

under suitable assumptions.

3. Consider

$$\begin{aligned} &u_t - \Delta u = f(u), && \text{in } \Omega\times\mathbf{R}^+,\\ &u = 0, && \text{in } \Gamma\times\mathbf{R}^+,\\ &u(\cdot,0) = v, && \text{in } \Omega. \end{aligned}$$

(a) Assume that $f(s) = -F'(s)$ with $F(s) \geq \frac{1}{2}s^2$ (a quadratic potential). Prove that

$$|u(t)|_1^2 + \|u(t)\|^2 \leq |v|_1^2 + 2\int_\Omega F(v)\,dx, \quad t\geq 0.$$

(b) Assume that f satisfies a global Lipschitz condition: $|f(s_1) - f(s_2)| \leq L|s_1 - s_2|$ for all $s_1, s_2 \in \mathbf{R}$. Show the stability estimate:

$$\|u_1(t) - u_2(t)\| \leq \mathrm{e}^{Lt}\|v_1 - v_2\|, \quad t \geq 0.$$

where u_1, u_2 are two solutions with different initial data v_1, v_2. Hint: Gronwall's lemma

$$\phi(t) \leq A + B\int_0^t \phi(s)\,\mathrm{d}s \text{ (with } B\geq 0) \quad \text{implies} \quad \phi(t)\leq A\mathrm{e}^{Bt}.$$

4. Consider the same equation as in Problem 3. (a) Formulate a semidiscrete finite element method (discretized only in the x-variables) for this problem.
(b) Prove an error estimate in the case when $f(u)$ is replaced by $f(x,t)$.
(c) Prove an error estimate in the case when $f(u)$ is globally Lipschitz as in Problem 3(b).

5. Consider the initial-boundary value problem for the wave equation

$$\begin{aligned} &u_{tt} - \Delta u = f, && x\in\Omega,\ t>0,\\ &u = 0, && x\in\Gamma,\ t>0,\\ &u(x,0) = v(x),\ u_t(x,0) = w(x), && x\in\Omega, \end{aligned}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded domain with smooth boundary Γ and $f = f(x,t)$. Show the following estimates by the energy method:

$$|u(t)|_1 + \|u_t(t)\| \leq C\Big(|v|_1 + \|w\| + \int_0^t \|f(s)\|\mathrm{d}s\Big),$$

$$\|u_{tt}(t)\| \leq C\Big(\|\Delta v\| + |w|_1 + \int_0^t \|f_t(s)\|\mathrm{d}s\Big).$$

Hint: $\int_0^t |(f,u_t)|\,\mathrm{d}s \leq \int_0^t \|f(s)\|\,\mathrm{d}s \max_{0\leq s\leq t}\|u_t(s)\|$. Take $f=0$ if you cannot do $f\neq 0$.

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TMA026 Partiella differentialekvationer, fk, TM, 2005–10–19 fm M. Solutions.

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1. See the book, Theorem A.4.

2. See the book, Section 3.5.

3. Weak formulation:

$$\begin{aligned}&u(t) \in H_0^1, \quad u(0) = v,\\ &(u_t, \phi) + a(u, \phi) = (f(u), \phi), \quad \forall \phi \in H_0^1,\ t > 0.\end{aligned}$$

(a) Choose $\phi = u_t(t)$:

$$\begin{aligned}&(u_t, u_t) + a(u, u_t) = (f(u), u_t) = -(F'(u), u_t)\\ &\|u_t\|^2 + \tfrac{1}{2} D_t |u|_1^2 + D_t \int_\Omega F(u)\,dx = 0\\ &\int_0^t \|u_t\|^2\,ds + \tfrac{1}{2}|u(t)|_1^2 + \int_\Omega F(u)\,dx = \tfrac{1}{2}|v|_1^2 + \int_\Omega F(v)\,dx\\ &\int_0^t \|u_t\|^2\,ds + \tfrac{1}{2}|u(t)|_1^2 + \tfrac{1}{2}\int_\Omega u(t)^2\,dx \le \tfrac{1}{2}|v|_1^2 + \int_\Omega F(v)\,dx\\ &|u(t)|_1^2 + \|u(t)\|^2 \le |v|_1^2 + 2\int_\Omega F(v)\,dx\end{aligned}$$

(b) Form the difference of the two equations, set $w = u_1 - u_2$, choose $\phi = w(t)$:

$$\begin{aligned}&(w_t, w) + a(w, w) = (f(u_1) - f(u_2), w)\\ &\tfrac{1}{2} D_t\|w\| + |w|_1^2 \le \|f(u_1) - f(u_2)\|\,\|w\| \le L\|u_1 - u_2\|\,\|w\| = L\|w\|^2\\ &\|w\| D_t\|w\| + |w|_1^2 = L\|w\|^2\\ &\|w\| D_t\|w\| \le L\|w\|^2\\ &D_t\|w\| \le L\|w\|\\ &\|w(t)\| \le \|w(0)\| + L\int_0^t \|w\|\,dt\\ &\|w(t)\| \le e^{Lt}\|w(0)\|\\ &\|u_1(t) - u_2(t)\| \le e^{Lt}\|v_1 - v_2\|\end{aligned}$$

4. (a)

$$\begin{aligned}&u_h(t) \in S_h, \quad u_h(0) = v_h,\\ &(u_{h,t}, \chi) + a(u_h, \chi) = (f(u_h), \chi), \quad \forall \chi \in S_h,\ t > 0.\end{aligned}$$

where S_h is the usual family of piecewis linear finite element spaces based on a family of triangulations of Ω, and $v_h \in S_h$ is some approximation of v.

(b) This is Theorem 10.1.

(c) Now equation (10.15) in the book becomes

$$(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi) + (f(u_h) - f(u), \chi) \quad \forall \chi \in S_h$$

Choose $\chi = \theta(t)$ and use the Lipschitz condition and Gronwall's lemma:

$$\tfrac{1}{2} D_t\|\theta\|^2 + |\theta|_1^2 \le \|\rho_t\|\|\theta\| + \|f(u_h) - f(u)\|\|\theta\| \le \|\rho_t\|\|\theta\| + L\|u_h - u\|\|\theta\|$$
$$\|\theta\| D_t\|\theta\| \le \|\rho_t\|\|\theta\| + L\|u_h - u\|\|\theta\|$$
$$D_t\|\theta\| \le \|\rho_t\| + L\|u_h - u\| \le \|\rho_t\| + L\|\theta\| + L\|\rho\|$$
$$\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \big(\|\rho_t\| + L\|\rho\|\big)\,ds + L\int_0^t \|\theta\|\,ds$$
$$\|\theta(t)\| \le \|\theta(0)\| + \int_0^T \big(\|\rho_t\| + L\|\rho\|\big)\,ds + L\int_0^t \|\theta\|\,ds, \quad t \in [0,T]$$
$$\|\theta(t)\| \le e^{Lt}\Big(\|\theta(0)\| + \int_0^T \big(\|\rho_t\| + L\|\rho\|\big)\,ds\Big), \quad t \in [0,T]$$
$$\|\theta(t)\| \le e^{Lt}\Big(\|\theta(0)\| + \int_0^t \big(\|\rho_t\| + L\|\rho\|\big)\,ds\Big), \quad t \ge 0$$

This leads to

$$\|u_h(t) - u(t)\| \le e^{Lt}\Big(\|v_h - v\| + Ch^2\|v\|_2 + Ch^2\int_0^t \big(\|u_t\|_2 + L\|u\|_2\big)\,ds\Big), \quad t \ge 0$$

in the same way as in Theorem 10.1.

5. Weak formulation:

$$u(t) \in H_0^1, \quad u(0) = v,\ u_t(0) = w,$$
$$(u_{tt}, \phi) + a(u, \phi) = (f, \phi), \quad \forall \phi \in H_0^1,\ t > 0.$$

(a) Choose $\phi = u_t(t)$.

$$(u_{tt}, u_t) + a(u, u_t) = (f, u_t)$$
$$\tfrac{1}{2} D_t\|u_t\|^2 + \tfrac{1}{2} D_t|u|_1^2 \le \|f\|\|u_t\|$$
$$\|u_t(t)\|^2 + |u(t)|_1^2 \le \|w\|^2 + |v|_1^2 + 2\int_0^t \|f\|\|u_t\|\,ds$$
$$\|u_t(t)\|^2 + |u(t)|_1^2 \le \|w\|^2 + |v|_1^2 + 2\int_0^t \|f(s)\|\,ds \max_{0\le s\le t}\|u_t(s)\|$$
$$\|u_t(t)\|^2 + |u(t)|_1^2 \le \|w\|^2 + |v|_1^2 + 4\Big(\int_0^t \|f(s)\|\,ds\Big)^2 + \tfrac{1}{4}\max_{0\le s\le t}\|u_t(s)\|^2$$
$$\|u_t(t)\|^2 + |u(t)|_1^2 \le \|w\|^2 + |v|_1^2 + 4\Big(\int_0^T \|f(s)\|\,ds\Big)^2 + \tfrac{1}{4}\max_{0\le s\le T}\|u_t(s)\|^2, \quad t \in [0,T]$$
$$\max_{0\le t\le T}\|u_t(t)\|^2 + \max_{0\le t\le T}|u(t)|_1^2 \le 2\max_{0\le t\le T}\Big(\|u_t(t)\|^2 + |u(t)|_1^2\Big)$$
$$\le 2\|w\|^2 + 2|v|_1^2 + 8\Big(\int_0^T \|f(s)\|\,ds\Big)^2 + \tfrac{1}{2}\max_{0\le s\le T}\|u_t(s)\|^2$$
$$\tfrac{1}{2}\max_{0\le t\le T}\|u_t(t)\|^2 + \max_{0\le t\le T}|u(t)|_1^2 \le 2\|w\|^2 + 2|v|_1^2 + 8\Big(\int_0^T \|f(s)\|\,ds\Big)^2$$
$$\max_{0\le t\le T}\|u_t(t)\|^2 + \max_{0\le t\le T}|u(t)|_1^2 \le C\Big(\|w\|^2 + |v|_1^2 + \Big(\int_0^T \|f(s)\|\,ds\Big)^2\Big)$$
$$\|u_t(t)\| + |u(t)|_1 \le C\Big(\|w\| + |v|_1 + \int_0^t \|f(s)\|\,ds\Big)$$

(b) Differentiate the equation with respect to t and choose $\phi = u_{tt}(t)$ and finish as in (a).

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