Matematik Chalmers

Tentamen i TMA026 Partiella differentialekvationer, fk, TM, 2006–08–28 f V

Telefon: Christoffer Cromvik 0762–721860
Inga hjälpmedel. Kalkylator ej tillåten.

You may get up to 10 points for each problem plus points for the hand-in problems.

Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Prove the inequality

$$\|v\|^2_{L_2(\Omega)} \le C\Big(\|\nabla v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Gamma)}\Big) \quad \forall v \in H^1(\Omega). \tag{1}$$

Here Ω is a bounded domain in $\mathbf{R}^d$ with smooth boundary Γ. (Do it for a square domain in the plane if you cannot do the general case.)

2. Give a weak formulation and prove existence and uniqueness of solutions to the boundary value problem

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f, && \text{in } \Omega,\\ a\frac{\partial u}{\partial n} + h(u-g) &= 0, && \text{on } \Gamma, \end{aligned}$$

under suitable assumptions on the coefficients a, h and the data functions f, g. Hint: use (1).

3. Consider

$$\begin{aligned} &u_t - \Delta u = f, && \text{in } \Omega\times\mathbf{R}^+,\\ &u = 0, && \text{on } \Gamma\times\mathbf{R}^+,\\ &u(\cdot,0) = v, && \text{in } \Omega. \end{aligned}$$

(a) Prove that

$$|u(t)|_1^2 + \int_0^t \big(\|u_t(s)\|^2 + \|\Delta u(s)\|^2\big)\,ds \le C\Big(|v|_1^2 + \int_0^t \|f(s)\|^2\,ds\Big), \quad t \ge 0.$$

(b) Assume $f = 0$. Prove that

$$\int_0^t s\|u_t(s)\|^2\,ds \le C\|v\|^2, \quad t\ge 0.$$

4. Consider the same equation as in Problem 3. (a) Formulate a semidiscrete finite element method (discretized only in the x-variables) for this problem.

(b) Prove an error estimate.

5. State and prove the maximum principle for the heat equation.

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TMA026 Partiella differentialekvationer, fk, TM, 2006–08–28 f V. Solutions.

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1. Integrate by parts in the identity $\int_\Omega v^2\,dx = \int_\Omega v^2\Delta\phi\,dx$, where $\phi(x) = \frac{1}{2d}|x|^2$.

2. Assume

$$a(x) \geq a_0 > 0, \quad h(x) \geq h_0 > 0$$

Assume $f \in L_2(\Omega)$, $g \in L_2(\Gamma)$. Weak formulation: Find $u \in H^1$ such that

$$a(u,v) = L(v) \quad \forall v \in H^1$$

where

$$\begin{aligned} a(v,w) &= \int_\Omega a\nabla v\cdot\nabla w\,dx + \int_\Gamma hvw\,ds \\ L(v) &= \int_\Omega fv\,dx + \int_\Gamma hgv\,ds \end{aligned}$$

Coercivity:

$$\begin{aligned} a(v,v) &= \int_\Omega a\nabla v\cdot\nabla v\,dx + \int_\Gamma hv^2\,ds \\ &\geq a_0\int_\Omega |\nabla v|^2\,dx + h_0\int_\Gamma v^2\,ds \\ &\geq \frac{1}{2}a_0\int_\Omega |\nabla v|^2\,dx + \min(\tfrac{1}{2}a_0, h_0)\Big(\int_\Omega |\nabla v|^2\,dx + \int_\Gamma v^2\,ds\Big) \\ &\geq \frac{1}{2}a_0\int_\Omega |\nabla v|^2\,dx + C\int_\Omega |v|^2\,dx \\ &\geq C\Big(\int_\Omega |\nabla v|^2\,dx + \int_\Omega |v|^2\,dx\Big) \\ &= C\|v\|_1^2 \quad \forall v \in H^1 \end{aligned}$$

Boundedness:

$$\begin{aligned} |a(v,w)| &\leq \max\big(\|a\|_{L_\infty}, \|h\|_{L_\infty}\big)\Big(\|v\|_1\|w\|_1 + \|v\|_{L_2(\Gamma)}\|w\|_{L_2(\Gamma)}\Big) \quad \text{(trace inequality)} \\ &\leq C\|v\|_1\|w\|_1 \end{aligned}$$

Similarly:

$$\begin{aligned} |L(v)| &\leq \|f\|\,\|v\| + C\|g\|_{L_2(\Gamma)}\|v\|_{L_2(\Gamma)} \\ &\leq C\Big(\|f\| + \|g\|_{L_2(\Gamma)}\Big)\|v\|_1 \end{aligned}$$

Existence and uniqueness now follows from Riesz's representation theorem.

3. Weak formulation:

$$\begin{aligned} &u(t) \in H_0^1, \quad u(0) = v, \\ &(u_t,\phi) + a(u,\phi) = (f,\phi), \quad \forall\phi \in H_0^1,\ t > 0. \end{aligned}$$

(a) Choose $\phi = u_t(t)$:

$$\begin{aligned}
&(u_t, u_t) + a(u, u_t) = (f, u_t) \\
&\|u_t\|^2 + \tfrac{1}{2} D_t |u|_1^2 \le \|f\| \, \|u_t\| \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_t\|^2 \\
&\int_0^t \|u_t\|^2 \, ds + |u(t)|_1^2 \le |v|_1^2 + \int_0^t \|f\|^2 \, ds
\end{aligned}$$

Then use also

$$\int_0^t \|\Delta u\|^2 \, ds = \int_0^t \|u_t - f\|^2 \, ds \le C\Big(\int_0^t \|u_t\|^2 \, ds + \int_0^t \|f\|^2 \, ds \Big)$$

(b) Choose $\phi = t u_t(t)$ and also $\phi = u(t)$.

4. (a)

$$\begin{aligned}
&u_h(t) \in S_h, \quad u_h(0) = v_h, \\
&(u_{h,t}, \chi) + a(u_h, \chi) = (f, \chi), \quad \forall \chi \in S_h, \ t > 0.
\end{aligned}$$

where S_h is the usual family of piecewis linear finite element spaces based on a family of triangulations of Ω, and $v_h \in S_h$ is some approximation of v.
(b) This is Theorem 10.1.

5. See the book.

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