Matematik Chalmers

Tentamen i TMA026 Partiella differentialekvationer, fk, TM, 2006–10–24 e V

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You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. (a) Prove the inequality

$$\|v\|_{L_1(\Gamma)} \le C\|v\|_{W_1^1(\Omega)}, \quad \forall v \in \mathcal{C}^1(\bar{\Omega}).$$

Here Ω is the unit square in $\mathbf{R}^2$ with boundary Γ. Also: $\|v\|_{L_1(\Gamma)} = \int_\Gamma |v|\,ds$ and $\|v\|_{W_1^1(\Omega)} = \|v\|_{L_1(\Omega)} + \|\nabla v\|_{L_1(\Omega)}$.
(b) Use this to construct a trace operator γ.

2. (a) Formulate the finite element method for

$$\begin{aligned} -\nabla\cdot(a\nabla u) &= f, && \text{in } \Omega,\\ u &= 0, && \text{on } \Gamma. \end{aligned}$$

(b) Prove error an estimate in the energy norm.
(c) Prove error an estimate in the L_2-norm.

3. Consider

$$\begin{aligned} u_t - \Delta u &= f, && \text{in } \Omega\times\mathbf{R}^+,\\ u &= 0, && \text{on } \Gamma\times\mathbf{R}^+,\\ u(\cdot,0) &= v, && \text{in } \Omega. \end{aligned}$$

(a) Use the energy method to prove that

$$|u(t)|_1^2 + \int_0^t \big(\|u_t(s)\|^2 + \|\Delta u(s)\|^2\big)\,ds \le C\Big(|v|_1^2 + \int_0^t \|f(s)\|^2\,ds\Big), \quad t \ge 0.$$

(b) Assume $f = 0$. Use the energy method to prove that

$$\|u_t(t)\| \le Ct^{-1}\|v\|, \quad t > 0.$$

4. Prove that

$$v \in H_0^1(\Omega) \quad \Longleftrightarrow \quad \sum_{j=1}^{\infty} \lambda_j (v,\varphi_j)^2 < \infty.$$

Here λ_j and φ_j are the eigenvalues and corresponding normalized eigenfunctions of the Laplace operator with Dirichlet boundary conditions as in Chapter 6 of the book.

5. State and prove the maximum principle for Poisson's equation $-\Delta u = f$.

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1. (a) Start with

$$v(0,x_2) = v(x_1,x_2) - \int_0^{x_1} \frac{\partial v}{\partial x_1}(y,x_2)\,dy.$$

(b) In a similar way as in the trace theorem in the book we can define $\gamma : W_1^1(\Omega) \to L_1(\Gamma)$.

2. See the book.

3. Weak formulation:

$$\begin{aligned} &u(t) \in H_0^1, \quad u(0) = v, \\ &(u_t,\phi) + a(u,\phi) = (f,\phi), \quad \forall \phi \in H_0^1,\ t > 0. \end{aligned}$$

(a) Choose $\phi = u_t(t)$:

$$\begin{aligned} &(u_t,u_t) + a(u,u_t) = (f,u_t) \\ &\|u_t\|^2 + \tfrac{1}{2} D_t |u|_1^2 \le \|f\|\,\|u_t\| \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_t\|^2 \\ &\int_0^t \|u_t\|^2\,ds + |u(t)|_1^2 \le |v|_1^2 + \int_0^t \|f\|^2\,ds \end{aligned}$$

Then use also

$$\int_0^t \|\Delta u\|^2\,ds = \int_0^t \|u_t - f\|^2\,ds \le C\Big(\int_0^t \|u_t\|^2\,ds + \int_0^t \|f\|^2\,ds\Big)$$

(b) Differentiate the equation with respect to t and choose $\phi = t^2 u_t(t)$.

4. See the book.

5. See the book.

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