Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2015–08–28 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Consider the unit square, $\Omega = [0,1] \times [0,1]$.

(a) Compute $\|x_1^2\|_{L^2(\Omega)}$, where x_1 is the first component of an element in $\mathbb{R}^2$.
(b) Show that $\|v\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}$, for all $v \in H_0^1(\Omega)$.
(c) Show that $\|v\|_{H^{-1}(\Omega)} \leq \|v\|_{L^2(\Omega)}$, for all $v \in L^2(\Omega)$.

2. Let $\Omega \subset \mathbb{R}^d$ be convex, with boundary Γ. Let $b \in \mathbb{R}^d$ be a constant vector and consider,

$$\begin{cases} -\Delta u + b \cdot \nabla u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma. \end{cases}$$

(a) Show that the corresponding weak form is coercive.
(b) Let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise linear functions. Derive the finite element method using the space V_h.
(c) Derive an a priori bound for the error in the finite element approximation. Express explicitly the dependency on b in the bound.

3. Consider the eigenvalue problem: find $u \in H_0^1(\Omega)$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u + cu = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

where $c \in L^\infty(\Omega)$ is positive.

(a) Show that the eigenvalues λ are real and positive.
(b) Show that eigenfunctions corresponding to different eigenvalues are orthogonal both with respect to the $L^2(\Omega)$ scalar product and to the energy scalar product induced by the problem, $(\nabla v, \nabla w) + (cv, w)$.
(c) Bound the smallest eigenvalue in terms of the smallest eigenvalue to the Laplacian $-\Delta$ on Ω and the bounded function c.

4. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the heat equation,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \Gamma \times (0,T), \\ u(\cdot,0) = v, & \text{in } \Omega. \end{cases}$$

(a) Let $v \in L^2(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \leq Ct^{-1/2}\|v\|_{L^2(\Omega)}$, for $t > 0$.
(b) Let $v \in H_0^1(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}$, for $t > 0$.
(c) Formulate the Crank-Nicolson Galerkin finite element method for this problem.

5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with smooth boundary Γ. Consider the wave equation,

$$\begin{cases} \ddot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot,0) = v, \quad \dot{u}(\cdot,0) = w, & \text{in } \Omega, \end{cases}$$

where v and w are smooth. Show that the total energy of u is constant in time.

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