Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2016–05–31 f M

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Let $\Omega \in \mathbb{R}^d$ be a polygonal domain with boundary Γ. Let $\mathcal{T}_h$ be a shape regular triangulation of Ω and $S_h = \{v \in C(\Omega) : v|_T \text{ is linear } \forall T \in \mathcal{T}_h, v|_\Gamma = 0\} = \text{span}(\{\phi_i\}_{i\in\mathcal{N}})$.

(a) Let $d = 1$ and assume that the mesh is uniform i.e. $\text{diam}(T) = h$, for all $T \in \mathcal{T}_h$ (a set of line segments). Show that ϕ_1 has a weak derivative and compute it.

(b) Now consider all cases $d = 1, 2, 3$ and also assume that the mesh is quasi-uniform, i.e. there is a constant so that $h := \max_{T\in\mathcal{T}_h} \text{diam}(T) \leq C \min_{T\in\mathcal{T}_h} \text{diam}(T)$. For which values of $p \geq 1$ can $\|\nabla\phi_i\|_{L^p(\Omega)}$ be bounded independent of h (here we assume $h < 1$)?

2. Consider the Neumann problem on a bounded domain Ω: find u such that

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega, \\ \partial_n u = 0, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$. Show that the problem has a unique weak solution.

3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary Γ. We consider the Helmholtz equation:

$$\begin{cases} -\Delta u - \omega^2 u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$ and $\omega \in \mathbb{R}$. Furthermore, let $\{\varphi_i\}_{i=1}^\infty$ be the orthonormal eigenfunctions with corresponding eigenvalues $\{\lambda_i\}_{i=1}^\infty$ fulfilling $(\nabla\varphi_i, \nabla v) = \lambda_i(\varphi_i, v)$ for all $v \in H_0^1(\Omega)$.

(a) Write the solution u as a linear combination of the eigenfunctions. For which values of ω is this representation valid?

(b) Let $\omega^2 < \lambda_1/2$. Show coercivity of $(\nabla v, \nabla v) - \omega^2(v, v)$ in the $|v|_{H^1(\Omega)}$ norm.

4. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the heat equation with $f, v \in L^2(\Omega)$,

$$\begin{cases} \dot{u} - \Delta u = f, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

(a) Formulate the backward Euler-Galerkin method for this problem with time step k.

(b) Show that the approximation U_n (of $u(t_n)$) is bounded in $L^2(\Omega)$.

(c) How should k and h (quasi uniform mesh) be related to balance the errors contributions?

5. Consider the symmetric hyperbolic system with smooth initial data v and forcing f,

$$\frac{\partial u}{\partial t} + \sum_{j=1}^d A_j \frac{\partial u}{\partial x_j} = f, \quad \text{in } \mathbb{R}^d \times \mathbb{R}^+.$$

Let v and f be N vectors and A_j *constant* symmetric $N \times N$ matrices and assume that u vanishes as $|x| \to \infty$.

(a) Show that $\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|v\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d)}\, ds$.

(b) Let $d = N = 1$. Compute the solution u using the method of characteristics.

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