Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2014–05–27 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ. Consider Laplace equation with $g \in C^2(\Omega)$,

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \Gamma. \end{cases}$$

(a) Derive the weak form. *Hint: Let $u = u_0 + g$ and seek $u_0 \in H_0^1(\Omega)$.*
(b) Bound $\|u\|_{L^\infty(\Omega)}$ (the max norm) using the maximum principle.
(c) Formulate the finite element method with boundary data $g_h = I_h g$ and derive an error estimate in energy norm $\|\nabla(u - u_h)\|$.

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ, and $I = (0, T)$. Consider the initial value problem,

$$(1) \qquad \begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases}$$

with $v \in L^2(\Omega)$.

(a) Express the solution in terms of the eigenfunctions and eigenvalues of $-\Delta$.
(b) Show that the $L^2(\Omega)$ norm of the solution decays exponentially in time.
(c) Assuming $v \in H_0^1(\Omega)$ show that

$$|u|_{H^1(\Omega)} \leq \min(Ct^{-1/2}\|v\|_{L^2(\Omega)}, |v|_{H^1(\Omega)}).$$

3. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with boundary Γ, and let $I = (0, T)$. Consider the semi-linear parabolic problem,

$$(2) \qquad \begin{cases} \dot{u} - \Delta u = f(u) := u(1 - u), & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

Assume $\|v\|_{H^1(\Omega)} \leq R_0$.

(a) Let $\|u(t)\|_{H^1(\Omega)} \leq R$ and $\|w(t)\|_{H^1(\Omega)} \leq R$ for $0 \leq t \leq T$.
Show that,

$$\|f(u) - f(w)\|_{L^2(\Omega)} \leq C(R)\|u - w\|_{H^1(\Omega)}, \quad t \in [0, T].$$

Hint: The inequality $\|w\|_{L^p(\Omega)} \leq C\|w\|_{H^1(\Omega)}$ holds for $1 \leq p \leq 6$.

(b) The solution to equation (2) can be written using the parabolic solution operator $E(t)$ (the solution operator to equation (1)), in the following way,

$$u(t) = E(t)v + \int_0^t E(t - s) f(u(s))\, ds.$$

Let $Su(t) = E(t)v + \int_0^t E(t - s) f(u(s))\, ds$, $\mathcal{B} = \{w : \max_{0 \leq t \leq \tau} \|w(t)\|_{H^1(\Omega)} \leq R\}$, and show that $S : \mathcal{B} \to \mathcal{B}$ for sufficiently large R and small τ.

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(c) Show that S is also a contraction mapping (and therefore has a fixed point $u = Su$), i.e. show

$$\max_{t\in[0,\tau]} \|Su - Sw\|_{H^1(\Omega)} \leq \gamma \max_{t\in[0,\tau]} \|u - w\|_{H^1(\Omega)},$$

where $\gamma < 1$, for sufficiently small τ.

4. Consider the following abstract elliptic problem in weak form: find $u \in H_0^1(\Omega)$ such that,

$$a(u, v) = l(v),$$

where a is a bilinear form, l is a linear functional, and Ω is a bounded domain.

(a) Show that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$. The trace theorem for functions in $H^1(\Omega)$ can be used without proof.

(b) Give sufficient assumptions on a and l so that the problem has a unique solution in $H_0^1(\Omega)$.

(c) Let $\Omega \subset \mathbb{R}^2$. Give an example of a non-constant b so that the bilinear form $a(u, v) = (\nabla u, \nabla v) + (b \cdot \nabla u, v)$ fulfills the conditions in (b).

5. Let $\Omega \subset \mathbb{R}^2$ be convex with smooth boundary, $v \in L^2(\Omega)$, and $f(t) \in L^2(\Omega)$, for $0 \leq t \leq T$. Show that the L^2-error in the semi-discrete Galerkin finite element approximation of the parabolic problem,

$$\begin{cases} \dot{u} - \Delta u &= f, \quad \text{in } \Omega \times (0, T) \\ u &= 0, \quad \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) &= v, \quad \text{in } \Omega, \end{cases}$$

is bounded in the following way:

$$\|u_h(t) - u(t)\|_{L^2(\Omega)} \leq \|v - v_h\|_{L^2(\Omega)} + Ch^2 \left(\|v\|_{H^2(\Omega)} + \int_0^t \|u_t\|_{H^2(\Omega)}\, ds \right), \quad t > 0.$$

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