Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2014–08–29 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

1.

(a) Find $u \in H_0^1(\Omega)$ such that, $(\nabla u, \nabla v) + (cu, v) = (f, v)$, for all $v \in H_0^1(\Omega)$.

(b) Let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise linear functions defined on a triangulation of Ω. The finite element method reads: find $u_h \in V_h$ such that,

$$a(u_h, v) = (f, v), \quad \forall v \in V_h.$$

The Galerkin Orthogonality $a(u - u_h, v) = 0$ for all $v \in V_h$ follows directly and gives us,

$$\begin{aligned}
\|\nabla(u-u_h)\|_{L^2(\Omega)}^2 &\le a(u-u_h, u-u_h) \\
&= a(u-u_h, u-v) \\
&\le \|\nabla(u-u_h)\|_{L^2(\Omega)}\|\nabla(u-v)\| + \beta\|u-u_h\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)} \\
&\le (1 + C^2\beta)\|\nabla(u-u_h)\|_{L^2(\Omega)}\|\nabla(u-v)\|_{L^2(\Omega)},
\end{aligned}$$

where C is the Poincare-Friedrich constant and $v \in V_h$ is arbitrary. We conclude

$$\|\nabla(u-u_h)\|_{L^2(\Omega)} = (1 + C^2\beta) \inf_{v \in V_h} \|\nabla(u-v)\|_{L^2(\Omega)}.$$

(c) Yes, let $\lambda_1 > 0$ be the smallest eigenvalue of $-\Delta$. Then with $c = -\frac{\lambda_1}{2}$ we have,

$$a(v,v) = (\nabla v, \nabla v) - \frac{\lambda_1}{2}(v,v) \ge \frac{1}{2}(\nabla v, \nabla v) + \frac{\lambda_1}{2}(v,v) - \frac{\lambda_1}{2}(v,v) = \frac{1}{2}\|\nabla v\|_{L^2(\Omega)}^2,$$

i.e. a is coercive.

2.

(a) Let $\{\lambda_i, \phi_i\}_{i=1}^{\infty}$ be the set of eigenvalues and (orthonormal) eigenvectors of $-\Delta$. We note that $u(x,t) = \sum_{i=1}^{\infty} \alpha_i(t)\phi_i$ with $\alpha_i(0) = (v, \phi_i)$ and $\alpha_i(t) = (v, \phi_i)e^{-(\lambda_i+\gamma)t}$.

(b) Therefore,

$$\|u(\cdot, t)\|_{L^2(\Omega)}^2 = \sum_{i=1}^{\infty} |(v, \phi_i)|^2 e^{-2t(\lambda_i+\gamma)},$$

i.e. the norm decreases exponentially with increasing γ for fix t.

(c) We multiply by $\dot{u}$ and integrate in over Ω and $[0,t]$ to get,

$$\begin{aligned}
\int_0^t \|\dot{u}(s)\|_{L^2(\Omega)}^2\,ds + \frac{1}{2}\left(\|\nabla u(t)\|_{L^2(\Omega)}^2 + \gamma\|u(t)\|_{L^2(\Omega)}^2\right) - \frac{1}{2}\left(\|\nabla v\|_{L^2(\Omega)}^2 + \gamma\|v\|_{L^2(\Omega)}^2\right) \\
= \int_0^t (\dot{u}, \dot{u}) + (\nabla u, \nabla \dot{u}) + \gamma(u, \dot{u})\,ds \\
= 0.
\end{aligned}$$

Therefore $\int_0^t \|\dot{u}(s)\|_{L^2(\Omega)}^2\,ds \le \frac{1}{2}\left(\|\nabla v\|_{L^2(\Omega)}^2 + \gamma\|v\|_{L^2(\Omega)}^2\right)$.

3.

(a) We have $\|f(u) - f(w)\|_{L^2(\Omega)} \leq \|u - w\|_{L^2(\Omega)} + \|u^2 + uv + v^2\|_{L^3(\Omega)}\|u - w\|_{L^6(\Omega)} \leq (1 + C2R^2)\|u - w\|_{H^1(\Omega)}$.

(b) We have

$$\begin{aligned}\|Su(t)\|_{H^1(\Omega)} &\leq \|v\|_{H^1(\Omega)} + \int_0^t (t-s)^{-1/2}\|f(u(s))\|_{L^2(\Omega)}\,ds\\ &\leq R_0 + C\tau^{1/2}R,\end{aligned}$$

since $\|f(u)\|_{L^2(\Omega)} \leq \|f(u) - f(0)\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)} \leq C \cdot R$. Let $R = 2R_0$ and τ be small enough for $C\tau^{1/2}R \leq R_0$.

(c) We have,

$$\begin{aligned}\|Su - Sw\|_{H^1(\Omega)} &\leq \int_0^t (t-s)^{-1/2}\|f(u(s)) - f(w(s))\|_{L^2(\Omega)}\,ds\\ &\leq C(R)\tau^{1/2}\max_{0\leq t\leq\tau}\|u(t) - w(t)\|_{H^1(\Omega)}.\end{aligned}$$

By taking $\max_{0\leq t\leq\tau}$ and choosing τ small enough for $\tau^{1/2}C(R) < 1$ we prove that S is a contraction.

4.

(a) Let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise linear functions defined on a triangulation of Ω. The BE-FEM approximation fulfills: find $u_h^n \in V_h$ such that,

$$(u_h^n - u_h^{n-1}, v) + k(\nabla u_h^n, \nabla v) = 0, \quad \forall v \in V_h,$$

with k being the time-step, $u_h^0 = P_h v$ and $P_h : L^2(\Omega) \to V_h$ isthe L^2-projection.

(b) We get,

$$\|u_h^n\|_{L^2(\Omega)}^2 \leq (u_h^n, u_h^n) + k(\nabla u_h^n, \nabla u_h^n) = (u_h^{n-1}, u_h^n) \leq \|u_h^n\|_{L^2(\Omega)}\|u_h^{n-1}\|_{L^2(\Omega)},$$

i.e. $\|u_h^n\|_{L^2(\Omega)} \leq \|u_h^{n-1}\|_{L^2(\Omega)}$ and therefore $\|u_h^n\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$.

(c) For smooth data it holds,

$$\|u(n \cdot k) - u_h^n\|_{L^2(\Omega)} \leq C_1 h^2 + C_2 k.$$

5. See the proof of Theorem A.4 in *Partial differential equations with numerical methods* by Thomée and Larsson. `/axel`