Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2015–06–02 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

1.

(a) The weak derivative of $|x+2|$ is -1 for $-4 < x < -2$ and 1 for $-2 < x < 4$.

(b) We use polar coordinates and note that $v'(r) = \frac{2}{r\log(r^2)} = \frac{1}{r\log(r)}$. We change variables $y = \log(r)$ to get,

$$|v|^2_{H^1(\Omega)} = 2\pi \int_0^{1/2} \frac{1}{r\log(r)^2}\,dr = 2\pi \int_{-\infty}^{-\log(2)} y^{-2}e^{-y}e^{y}\,dy = 2\pi \int_{-\infty}^{-\log(2)} y^{-2}\,dy < \infty.$$

In $L^2(\Omega)$ norm the singularity is weaker so it is also bounded by similar calculation.

(c) $\|x\|_{H^{-1}(\Omega)} = \sup_{v\in H^1_0(\Omega)} \frac{\int_0^1 xv(x)\,dx}{\|v'\|_{L^2(\Omega)}} = \sup_{v\in H^1_0(\Omega)} \frac{-\int_0^1 \frac{x^2}{2}v'(x)\,dx + [\frac{x^2}{2}v(x)]_0^1}{\|v'\|_{L^2(\Omega)}} \le \|\frac{x^2}{2}\|_{L^2(\Omega)} = \frac{1}{2\sqrt{5}}$.

2.

(a) Find $\tilde{u} \in H^1_0(\Omega)$ such that $(\tilde{a}\nabla\tilde{u}, \nabla v) = (f, v)$, for all $v \in H^1_0(\Omega)$. Furthermore,

$a_0\|\nabla(u-\tilde{u})\|^2_{L^2(\Omega)} \le (a\nabla(u-\tilde{u}), \nabla(u-\tilde{u})) = ((\tilde{a}-a)\nabla\tilde{u}, \nabla(u-\tilde{u})) \le \|\tilde{a}-a\|_{L^\infty(\Omega)}\|\nabla\tilde{u}\|\|\nabla(u-\tilde{u})\|$.

Furthermore, by letting $v = \tilde{u}$ in the weak form we get $a_0\|\nabla\tilde{u}\|^2_{L^2(\Omega)} \le (\tilde{a}\nabla\tilde{u}, \nabla\tilde{u}) = (f, \tilde{u}) \le C\|f\|_{L^2(\Omega)}\|\nabla\tilde{u}\|$ with a Poincare constant C. We conclude,

$$\|\nabla(u-\tilde{u})\|_{L^2(\Omega)} \le Ca_0^{-2}\|\tilde{a}-a\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}.$$

(b) Find $\tilde{u}_h \in V_h$ such that $(\tilde{a}\nabla\tilde{u}_h, \nabla v) = (f, v)$, for all $v \in V_h$. For this formulation we have full Galerkin Orthogonality between $\tilde{u} - \tilde{u}_h$ and V_h. We get,

$$a_0\|\nabla(\tilde{u}-\tilde{u}_h)\|^2_{L^2(\Omega)} \le (\tilde{a}\nabla(\tilde{u}-\tilde{u}_h), \nabla(\tilde{u}-\tilde{u}_h)) \le a_1\|\nabla(\tilde{u}-\tilde{u}_h)\|\|\nabla(\tilde{u}-I_h\tilde{u})\|_{L^2(\Omega)},$$

i.e. $\|\nabla(\tilde{u}-\tilde{u}_h)\|_{L^2(\Omega)} \le C\frac{a_1}{a_0}h\|D^2\tilde{u}\|_{L^2(\Omega)}$. Using the triangle inequality we get,

$$\|\nabla(u-\tilde{u}_h)\|_{L^2(\Omega)} \le Ca_0^{-2}\|\tilde{a}-a\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)} + C\frac{a_1}{a_0}h\|D^2\tilde{u}\|_{L^2(\Omega)}.$$

(c) It may be computationally expensive or impossible to integrate the diffusion coefficient exactly. Then numerical quadrature is the only option. The quadrature error committed can be bounded with the analysis above.

3.

(a) We divide $(0, T)$ into N time steps of equal size $k = T/N$ and let $\bar{\partial}_t U^n = k^{-1}(U^n - U^{n-1})$. The backward-Euler Galerkin method reads, find $\{U^n\} \in V_h$ such that,

$$(\bar{\partial}_t U^n, w) + (\nabla U^n, \nabla w) = (f(t_n), w), \quad \forall v \in V_h,\ n \ge 1,$$
$$U^0 = v_h.$$

(b) Let $w = U^n$. Since $(\nabla U^n, \nabla U^n) \ge 0$,

$$(\bar{\partial}_t U^n, U^n) \le \|f^n\|_{L^2(\Omega)}\|U^n\|_{L^2(\Omega)} \le 0.$$

This means that,

$$\|U^n\|^2_{L^2(\Omega)} \le (U^n, U^{n-1}) \le \|U^n\|_{L^2(\Omega)}\|U^{n-1}\|_{L^2(\Omega)},$$

i.e. $\|U^n\|_{L^2(\Omega)}$ is decreasing as n increases.

(c) For smooth solution u the errors are bounded in the following way:

$$\|u - U^n_{\mathrm{BE}}\|_{L^2(\Omega)} \leq C_1 h^2 + C_2 k,$$
$$\|u - U^n_{\mathrm{CN}}\|_{L^2(\Omega)} \leq C_1 h^2 + C_2 k^2.$$

4.

(a) Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues with corresponding eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$. Let,

$$u(x,t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\phi_j(x).$$

We insert this into the wave equation and get,

$$\sum_{j=1}^{\infty} (\hat{u}_j''(t) + \lambda_j \hat{u}_j(t))\phi_j(x) = 0.$$

Correspondingly $v = u(x,0) = \sum_{j=1}^{\infty} \hat{u}_j(0)\varphi_j(x)$ and $w = u_t'(x,0) = \sum_{j=1}^{\infty} \hat{u}_j'(0)\varphi_j(x)$. Since the eigenfunctions are orthogonal in $L^2(\Omega)$ we conclude that,

$$\hat{u}_j''(t) + \lambda_j \hat{u}_j(t) = 0, \quad t > 0,$$
$$\hat{u}_j(0) = (v, \phi_j),$$
$$\hat{u}_j'(0) = (w, \phi_j).$$

Therefore,

$$\hat{u}_j(t) = (v, \phi_j)\cos(\lambda_j^{1/2} t) + \frac{(w, \phi_j)}{\lambda_j^{1/2}} \sin(\lambda_j^{1/2} t),$$

for $j \geq 1$, i.e.,

$$u(x,t) = \sum_{j=1}^{\infty} ((v, \phi_j)\cos(\lambda_j^{1/2} t) + \frac{(w, \phi_j)}{\lambda_j^{1/2}} \sin(\lambda_j^{1/2} t))\varphi_j.$$

(b) We multiply the equation by u_t and integrate in space to get,

$$0 = (u_{tt}, u_t) + (\nabla u, \nabla u_t) = \frac{1}{2}\frac{\partial}{\partial t}(u_t, u_t) + \frac{1}{2}\frac{\partial}{\partial t}(\nabla u, \nabla u) := \frac{1}{2}\frac{\partial}{\partial t}\mathcal{E}(t),$$

for all $t > 0$.

5. See proof of Theorem 3.1 in Larsson and Thomeé *Partial differential equations with numerical methods,* 2005.

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