Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2016–08–26 f M

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Consider the Poisson equation $-\Delta u = f$ in $\mathbb{R}^3$.

(a) Show that the fundamental solution $U(x) = \frac{1}{4\pi|x|}$.
(b) Show that $u(x) = (U * f)(x) = \int_{\mathbb{R}^3} U(x-y) f(y)\, dy$.

2. Consider the Neumann problem, find u such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \partial_n u = g, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

(a) Under what additional assumption on f and g do we have existence of solution?
(b) Show that a solution u can not be unique.
(c) What is the smallest eigenvalue of the corresponding eigenvalue problem, where $g = 0$ and f is replaced by λu?

3. Consider the following abstract elliptic problem in weak form: find $u \in H_0^1(\Omega)$ such that,

$$a(u, v) = l(v),$$

where a is a bilinear form, l is a linear functional, and Ω is a bounded domain in $\mathbb{R}^3$.

(a) Show that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$. The trace theorem for functions in $H^1(\Omega)$ can be used without proof.
(b) Give sufficient assumptions on a and l so that the problem has a unique solution in $H_0^1(\Omega)$.
(c) Give an example of a linear functional l that violates the conditions in (b).

4. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the heat equation,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \Gamma \times (0,T), \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases}$$

(a) Let $v \in L^2(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \le C t^{-1/2} \|v\|_{L^2(\Omega)}$, for $t > 0$.
(b) Let $v \in H_0^1(\Omega)$. Show that $\|\nabla u(t)\|_{L^2(\Omega)} \le \|\nabla v\|_{L^2(\Omega)}$, for $t > 0$.
(c) Formulate the Crank-Nicolson-Galerkin method for this problem.

5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with smooth boundary Γ. Consider the wave equation,

$$\begin{cases} \ddot{u} - \Delta u = f, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, \quad \dot{u}(\cdot, 0) = w, & \text{in } \Omega. \end{cases}$$

Let u_h be the semi-discrete (in space) Galerkin approximation of u using v_h and w_h as approximations for the initial conditions. Prove for $t \ge 0$ that,

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} \le C\left(|v_h - R_h v|_{H^1(\Omega)} + \|w_h - R_h w\|\right) + Ch^2 \left(\|u(t)\|_{H^2(\Omega)} + \int_0^t \|u_{tt}\|_{H^2(\Omega)}\, ds\right),$$

where R_h is the Ritz projection.

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