Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2017–08–23 f SB

Telefon: Sebastian Jobjörnsson 031–7725325
Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Consider the Poisson equation in one dimension on the unit interval $\Omega = (0, 1)$: find $u(x)$ such that

$$\begin{cases} -(a(x)u'(x))' = f(x), & \text{in } \Omega, \\ u(0) = u(1) = 0, \end{cases}$$

where $f \in L^2(\Omega)$ and $a \in C^1(\Omega)$ such that $a \geq a_0 > 0$.

(a) Write the problem on weak form and bound the $H^1(\Omega)$ norm of u in terms of data.
(b) Bound the $H^2(\Omega)$ semi-norm of u in terms of data.
(c) Assume now that a is rapidly varying in space, i.e. $a'(x) \approx \epsilon^{-1}$ for some small ϵ. How does this affect the H^1 and H^2 norms respectively and how does it affects the approximation error between the solution u and its interpolant $I_h u$ computed on a uniform mesh of mesh size h?

2. Consider the convection-diffusion equation on a bounded convex domain $\Omega \subset \mathbb{R}^2$ with boundary Γ: find u such that,

$$\begin{cases} -\epsilon\Delta u + b \cdot \nabla u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

where $b = [1, 0]^T$, $f \in L^2(\Omega)$, and ϵ is a given (small) number.

(a) Show existence and uniqueness of weak solution using the Lax-Milgram Lemma.
(b) Bound the $H^1(\Omega)$ and the $H^2(\Omega)$ semi norms in terms of f and ϵ.

3. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary Γ. Consider the eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

(a) Write the problem on weak form and derive the finite element approximation using continuous piecewise linear basis functions on a triangulation.
(b) Show that the discrete eigenvalues are greater than or equal to the exact ones. The min-max principles can be used without proof.
(c) What is the convergence order for the eigenvalue error if the problem is posed on a convex domain (no proof is needed)?

4. State and prove the parabolic maximum principle.

5. Let $\Omega \subset \mathbb{R}^3$ be a domain, with boundary Γ. Let u solve,

$$\begin{cases} \dot{u} - \Delta u = f(u) := u - u^3, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \Gamma \times (0, T), \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases}$$

(a) Show that for any $u, v \in \{w \in H^1(\Omega) : \|w\|_{H_0^1(\Omega)} < R\}$ it holds $\|f(u) - f(v)\|_{L^2(\Omega)} \leq C(R)\|u - v\|_{H^1(\Omega)}$.
(b) *A priori boundedness:* Show that any solution u can be bounded in the $H^1(\Omega)$-norm for all $t > 0$ in terms of the initial data v.

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