Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2017–05–30 f SB

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Sol. See Theorem A.4 in Thomée-Larsson.

2.

(a) Find $u \in H^1(\Omega)$ such that

$$a(u,v) := (\nabla u, \nabla v) + \kappa(u,v)_\Gamma = (f,v), \quad \forall v \in H^1(\Omega).$$

(b) We have

$$\|v\|^2_{H^1(\Omega)} \le (1+C^2)\|\nabla v\|^2_{L^2(\Omega)} + C^2\kappa^{-1}\kappa\|v\|^2_{L^2(\Gamma)} \le \max(1+C^2, C^2\kappa^{-1})a(v,v),$$

and $a(v,w) \le (1+C^2\kappa)\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}$, using the trace inequality.

(c) Lets call the coercivity constant $\alpha = (\max(1+C^2, C^2\kappa^{-1}))^{-1}$. We then get,

$$\alpha\|u\|^2_{H^1(\Omega)} \le a(u,u) = (f,u) \le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)},$$

i.e. $\|u\|_{H^1(\Omega)} \le \alpha^{-1}\|f\|_{L^2(\Omega)}$.

3. Sol.

(a) Let $\lambda_i \neq \lambda_j$. We get

$$(\lambda_i - \lambda_j)(\phi_i, \phi_j) = a(\phi_i, \phi_j) - a(\phi_j, \phi_i) = 0,$$

with $a(\phi_i, \phi_j) = (\nabla\phi_i, \nabla\phi_j) + (V\phi_i, \phi_j)$ which is symmetric. Since $\lambda_i \neq \lambda_j$ by assumption ϕ_i and ϕ_j are orthogonal in the L^2 scalar product. We also have $\lambda_i\|\phi_i\|^2_{L^2(\Omega)} = \|\nabla\phi_i\|^2_{L^2(\Omega)} + \|V^{1/2}\phi_i\|^2_{L^2(\Omega)} \in \mathbb{R}^+$ and $\|\phi_i\|^2_{L^2(\Omega)} \in \mathbb{R}^+$ i.e. λ_i is real and positive for all i.

(b) Let the mimimum be realized at $u = u_1 \in H^1_0(\Omega)$. We get,

$$\lambda_1 = \frac{\|\nabla u_1\|^2_{L^2(\Omega)} + (Vu_1, u_1)}{\|u_1\|^2_{L^2(\Omega)}} \ge \frac{\|\nabla u_1\|^2_{L^2(\Omega)}}{\|u_1\|^2_{L^2(\Omega)}} \ge C^{-1} > 0,$$

where we have used that $V \ge 0$ and the Poincaré inequality $\|u_1\|_{L^2(\Omega)} \le C\|\nabla u_1\|_{L^2(\Omega)}$, since $u_1 \in H^1_0(\Omega)$.

4. Sol.

(a) We represent u using the orthonormal eigenfunctions of the negative Laplacian. We get $u = \sum_{i=1}^\infty e^{-\lambda_i t}(v,\phi_i)\phi_i$. Since $\{\phi_i\}_{i=1}^\infty$ is an orthonormal basis and $\lambda_i > 0$ we get $\|u\|^2_{L^2(\Omega)} = \sum_{i=1}^\infty e^{-2\lambda_i t}(v,\phi_i)^2 \le \sum_{i=1}^\infty (v,\phi_i)^2 = \|v\|^2_{L^2(\Omega)}$.

(b) Find $U^n \in V_h$, $1 \le n \le N$, such that,

$$(U^n, w) + k(\nabla U^n, \nabla w) = (U^{n-1}, w), \quad \forall w \in V_h.$$

(c) Let $w = U^n$. We get,

$$\|U^n\|^2_{L^2(\Omega)} + k\|\nabla U^n\|^2_{L^2(\Omega)} \le \|U^{n-1}\|_{L^2(\Omega)}\|U^n\|_{L^2(\Omega)} \le \frac{1}{2}\|U^{n-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|U^n\|^2_{L^2(\Omega)}.$$

Therefore $(1+2kC^{-2})\|U^n\|^2_{L^2(\Omega)} \le \|U^n\|^2_{L^2(\Omega)} + 2k\|\nabla U^n\|^2_{L^2(\Omega)} \le \|U^{n-1}\|^2_{L^2(\Omega)}$, where $C > 0$ is the Poincaré constant. We get $\|U^n\|^2_{L^2(\Omega)} \le (1+2kC^{-2})^{-1}\|U^{n-1}\|^2_{L^2(\Omega)} < \|U^{n-1}\|^2_{L^2(\Omega)}$ i.e. decreasing (assuming non trivial solution).

5. Sol.

(a) The nonlinear data f is evaluated at U^{n-1} instead of U^n. IMEX avoids solving a nonlinear system in each iteration.

(b) The next iterate $U^n \in V_h$ fulfills the elliptic equation $a(U^n, w) = l(w)$ with $a(v, w) = (v, w) + k(\nabla v, \nabla w)$ and $l(w) = (U^{n-1}, w) + k(f(U^{n-1}), w)$. We first show that a is coercive and bounded in H^1. We have $a(v, v) \geq \min(1, k)\|v\|^2_{H^1(\Omega)}$ and $a(v, w) \leq \max(1, k)\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}$. We turn to the linear functional. We have $|l(w)| \leq \|U^{n-1}\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)} + kB\|U^{n-1}\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}$. Given $U^0 \in V_h$ we get existence and uniqueness of U^1 by Lax-Milgram. Then we can continue to get existence for any iterate n.

(c) We have

$$\begin{aligned}\|U^n\|^2_{L^2(\Omega)} &\leq \|U^n\|^2_{L^2(\Omega)} + k\|\nabla U^n\|^2_{L^2(\Omega)} \\ &= (U^{n-1}, U^n) + k(f(U^{n-1}), U^n) \\ &\leq (1 + kB)\|U^{n-1}\|_{L^2(\Omega)}\|U^n\|_{L^2(\Omega)},\end{aligned}$$

or $\|U^n\|_{L^2(\Omega)} \leq (1 + \frac{TB}{N})^n \|v_h\|_{L^2(\Omega)} \leq e^{BT}\|v_h\|_{L^2(\Omega)}$ for all $1 \leq n \leq N$.

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