Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2018–05–29 f SB

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Consider the Neumann problem on a bounded domain Ω: find u such that

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega, \\ \partial_n u = 0, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$. Show that the problem has a unique weak solution which fulfills $\|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$.

2. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain. Consider the Poisson equation on weak form: find $u \in H_0^1(\Omega)$ such that, $(\nabla u, \nabla v) = (f, v)$ for all $v \in H_0^1(\Omega)$ where $f \in L^2(\Omega)$.

(a) Show that $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$.
(b) Show that $\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^{6/5}(\Omega)}$. *Hint:* $\|v\|_{L^6(\Omega)} \leq C'\|v\|_{H^1(\Omega)}$ *for all* $v \in H_0^1(\Omega)$.
(c) Show that $F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega fv\,dx$ is minimized (over $H_0^1(\Omega)$) by u.

3. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain, with boundary Γ, and let $I = (0, T)$. Consider the semi-linear parabolic problem,

$$\begin{cases} \dot{u} - \Delta u = f(u) := u - u^3, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{1}$$

where $v \in H_0^1(\Omega)$.

(a) Show that $f(u)$ fulfills $\|f(u) - f(v)\|_{L^2(\Omega)} \leq C(R)\|u - v\|_{H^1(\Omega)}$, for all $u, v \in B_R = \{w \in H_0^1(\Omega) : \|w\|_{H^1(\Omega)} \leq R\}$.
(b) Given a solution, which fulfills $u(t) \in H^1(\Omega)$ and $\dot{u}(t) \in L^2(\Omega)$ for a fix time t, show that $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$.
(c) Formulate the Backward Euler Galerkin method for equation (1) but with $f(U^n)$ replaced by $f(U^{n-1})$ (this is an implicit-explicit method). Show the existence of iterate U^n given U^{n-1}.

4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ, and $I = (0, T)$. Consider the initial value problem,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{2}$$

with $v \in L^2(\Omega)$.

(a) Formulate the Crank-Nicolson-Galerkin method for the problem.
(b) Show that the $L^2(\Omega)$ norm of the solution is bounded by the initial value for all $t \geq 0$.
(c) Assume u to be smooth in space and time. How does the error in $L^2(\Omega)$ norm for a fixed t depend on the mesh size and the time step?

Continued on page 2!

5. Prove the min-max principle for the n:th eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions, i.e.,

$$\lambda_n = \min_{V_n} \max_{v \in V_n} \frac{(\nabla v, \nabla v)}{(v, v)},$$

where V_n varies over all subspaces of $H_0^1(\Omega)$ of finite dimension n.

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