Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2018–05–29 f SB

Telefon: Axel Målqvist 031–7723599
Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Consider the Neumann problem on a bounded domain Ω: find u such that

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega, \\ \partial_n u = 0, & \text{on } \Gamma, \end{cases}$$

where $f \in L^2(\Omega)$. Show that the problem has a unique weak solution which fulfills $\|u\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$. **Sol.** Let $a(u,v) = (\nabla u, \nabla v) + (u,v)$ which is the scalar product in $H^1(\Omega)$. We therefore have $a(v,v) = \|v\|^2_{H^1(\Omega)}$ i.e. coercivity and $|a(u,v)| \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$ i.e. boundedness. Also $L(v) := (f,v) \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}$ i.e. bounded. Lax-Milgram guarantees existence and uniqueness. We directly have $\|u\|^2_{H^1(\Omega)} = a(u,u) = L(u) \leq \|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}$.

2. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain. Consider the Poisson equation on weak form: find $u \in H^1_0(\Omega)$ such that, $(\nabla u, \nabla v) = (f,v)$ for all $v \in H^1_0(\Omega)$ where $f \in L^2(\Omega)$.

(a) Show that $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$.
(b) Show that $\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^{6/5}(\Omega)}$. *Hint:* $\|v\|_{L^6(\Omega)} \leq C'\|v\|_{H^1(\Omega)}$ *for all* $v \in H^1_0(\Omega)$.
(c) Show that $F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega fv\,dx$ is minimized (over $H^1_0(\Omega)$) by u.

Sol. Let $v = u$ to get $\|\nabla u\|^2_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}$ using PF i.e. $\|\nabla u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ and thereby $\|u\|_{H^1(\Omega)} \leq C'\|f\|_{L^2(\Omega)}$ again by PF. Convex gives us $\|D^2 u\|_{L^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)} = C\|f\|_{L^2(\Omega)}$ i.e. $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. We also have $C^{-1}\|u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)} = (f,u) \leq \|fu\|_{L^1(\Omega)} \leq \|f\|_{L^{6/5}(\Omega)}\|u\|_{L^6(\Omega)} \leq C\|f\|_{L^{6/5}(\Omega)}\|u\|_{H^1(\Omega)}$ using PF, Hölder with $p = 6/5$ and $q = 6$, and Sobolev. Therefore $\|u\|_{H^1(\Omega)} \leq C\|f\|_{L^{6/5}(\Omega)}$. Finally let $v = u + w$ for any $w \in H^1_0(\Omega)$. We get

$$F(v) = F(u) + \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \int_\Omega \nabla u \cdot \nabla w\,dx - \int_\Omega fw\,dx = F(u) + \frac{1}{2}\int_\Omega |\nabla w|^2\,dx \geq F(u).$$

3. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain, with boundary Γ, and let $I = (0,T)$. Consider the semi-linear parabolic problem,

$$\begin{cases} \dot{u} - \Delta u = f(u) := u - u^3, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{1}$$

where $v \in H^1_0(\Omega)$.

(a) Show that $f(u)$ fulfills $\|f(u) - f(v)\|_{L^2(\Omega)} \leq C(R)\|u - v\|_{H^1(\Omega)}$, for all $u, v \in B_R = \{w \in H^1_0(\Omega) : \|w\|_{H^1(\Omega)} \leq R\}$.
(b) Given a solution, which fulfills $u(t) \in H^1(\Omega)$ and $\dot{u}(t) \in L^2(\Omega)$ for a fix time t, show that $u(t) \in H^2(\Omega) \cap H^1_0(\Omega)$.
(c) Formulate the Backward Euler Galerkin method for equation (1) but with $f(U^n)$ replaced by $f(U^{n-1})$ (this is an implicit-explicit method). Show the existence of iterate U^n given U^{n-1}.

Sol. We have $\|f(u) - f(w)\|_{L^2(\Omega)} \leq \|u - w\|_{L^2(\Omega)} + \|u^2 + uv + v^2\|_{L^3(\Omega)}\|u - w\|_{L^6(\Omega)} \leq (1 + C2R^2)\|u - w\|_{H^1(\Omega)}$.

Note that $g := f(u(t)) - \dot{u(t)} \in L^2(\Omega)$. Therefore u solves $-\Delta u(t) = g(t) \in L^2(\Omega)$ on a convex domain for the t in the problem. Elliptic regularity guarantees that $u(t) \in H^2(\Omega)$.

The next iterate $U^n \in V_h$ fulfills the elliptic equation $a(U^n, w) = l(w)$ with $a(v, w) = (v, w) + k(\nabla v, \nabla w)$ and $l(w) = (U^{n-1}, w) + k(f(U^{n-1}), w)$. We first show that a is coercive and bounded in H^1. We have $a(v, v) \geq \min(1, k)\|v\|^2_{H^1(\Omega)}$ and $a(v, w) \leq \max(1, k)\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}$. We turn to the linear functional. We have $|l(w)| \leq \|U^{n-1}\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)} + kB\|U^{n-1}\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}$. Given $U^0 \in V_h$ we get existence and uniqueness of U^1 by Lax-Milgram. Then we can continue to get existence for any iterate n.

4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ, and $I = (0, T)$. Consider the initial value problem,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{2}$$

with $v \in L^2(\Omega)$.

(a) Formulate the Crank-Nicolson-Galerkin method for the problem.
(b) Show that the $L^2(\Omega)$ norm of the solution is bounded by the initial value for all $t \geq 0$.
(c) Assume we have a problem with a smooth solution for all times discretized using the Crank-Nicolson-Galerkin method with continuous piecewise linear basis functions. Further assume we can evaluate the error in $L^2(\Omega)$ norm for a fixed time t. How will the error depend on the time-step k and the mesh parameter h respectively?

Sol. Let $V_h \subset H^1_0(\Omega)$ be the space of continuous piecewise linear functions defined on a triangulation of Ω. The CN-FEM approximation fulfills: find $u_h^n \in V_h$ such that,

$$(u_h^n - u_h^{n-1}, v) + \frac{k}{2}(\nabla(u_h^n + u_h^{n-1}), \nabla v) = 0, \quad \forall v \in V_h,$$

with k being the time-step, $u_h^0 = P_h v$ and $P_h : L^2(\Omega) \to V_h$ is the L^2-projection. We let $v = u_h^n + u_h^{n-1}$ and get $\|u_h^n\|_{L^2(\Omega)} \leq \|u_h^{n-1}\|_{L^2(\Omega)}$ and therefore $\|u_h^n\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}$ (see page 159 in Larsson-Thomée for details).

For smooth data it holds,

$$\|u(n \cdot k) - u_h^n\|_{L^2(\Omega)} \leq C_1 h^2 + C_2 k^2.$$

5. Prove the min-max principle for the n:th eigenvalue to the Laplace operator with homogeneous Dirichlet boundary conditions, i.e.,

$$\lambda_n = \min_{V_n} \max_{v \in V_n} \frac{(\nabla v, \nabla v)}{(v, v)},$$

where V_n varies over all subspaces of $H^1_0(\Omega)$ of finite dimension n. **Sol.** See Theorem 6.5 in Larsson- Thomée.

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