Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2018–08–25 f SB

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40– (Chalmers) G: 20–34p, VG: 35– (GU).

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1. Consider the following problems.

(a) Compute the weak derivative of $|x|$ on the interval $[-1, 1]$.
(b) Let $\Omega = \{x \in \mathbb{R}^2 : |x| \le 1/2\}$. Show that $v(x) = \log(-\log(|x|^2))$ belongs to $H^1(\Omega)$.
(c) Let $\Omega = \{x \in \mathbb{R}^d : |x| \le 1\}$. For which values $\lambda \in \mathbb{R}$ does $|x|^\lambda$ belong to $L^2(\Omega)$?

2. Let $\Omega \subset \mathbb{R}^d$ be bounded, with boundary Γ. Let $f \in L^2(\Omega)$ and consider,

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \Gamma. \end{cases}$$

(a) Formulate weak form and the finite element method.
(b) Show that the solution $u \in H_0^1(\Omega)$ minimizes the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_\Omega f v\, dx.$$

3. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ, and $I = (0, T)$. Consider the initial value problem,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{1}$$

with $v \in L^2(\Omega)$.

(a) Express the solution as $u(t) = E(t)v$ using the eigenpairs of the Laplacian and v.
(b) Show that the $L^2(\Omega)$ norm of the solution is bounded by the initial value for all $t \ge 0$.
(c) Formulate the Galerkin finite element method with Backward Euler time-stepping for the problem.

4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with boundary Γ, and let $I = (0, T)$. Consider the semi-linear parabolic problem,

$$\begin{cases} \dot{u} - \Delta u = f(u) := u - u^3, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases} \tag{2}$$

Assume $\|v\|_{H^1(\Omega)} \le R_0$.

(a) Let $\|u(t)\|_{H^1(\Omega)} \le R$ and $\|w(t)\|_{H^1(\Omega)} \le R$ for $0 \le t \le T$. Show that,

$$\|f(u) - f(w)\|_{L^2(\Omega)} \le C(R)\|u - w\|_{H^1(\Omega)}, \quad t \in [0, T].$$

Hint: The inequality $\|w\|_{L^p(\Omega)} \le C\|w\|_{H^1(\Omega)}$ holds for $1 \le p \le 6$.

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(b) The solution to equation (2) can be written using the parabolic solution operator $E(t)$ (the solution operator to equation (1)), in the following way,

$$u(t) = E(t)v + \int_0^t E(t-s)f(u(s))\,ds.$$

Let $Su(t) = E(t)v + \int_0^t E(t-s)f(u(s))\,ds$, $\mathcal{B} = \{w : \max_{0\le t\le\tau} \|w(t)\|_{H^1(\Omega)} \le R\}$, and show that $S : \mathcal{B} \to \mathcal{B}$ for sufficiently large R and small τ.

(c) Show that S is also a contraction mapping (and therefore has a fixed point $u = Su$), i.e. show

$$\max_{t\in[0,\tau]} \|Su - Sw\|_{H^1(\Omega)} \le \gamma \max_{t\in[0,\tau]} \|u - w\|_{H^1(\Omega)},$$

where $\gamma < 1$, for sufficiently small τ.

5. Let $\Omega = [0,1]^2$, $f \in L^2(\Omega)$, and let u solve

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma. \end{cases}$$

Show that the finite element approximation $u_h \in V_h \subset H_0^1(\Omega)$ of u fulfills the bound

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2 \|f\|_{L^2(\Omega)},$$

where h is the mesh size.

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