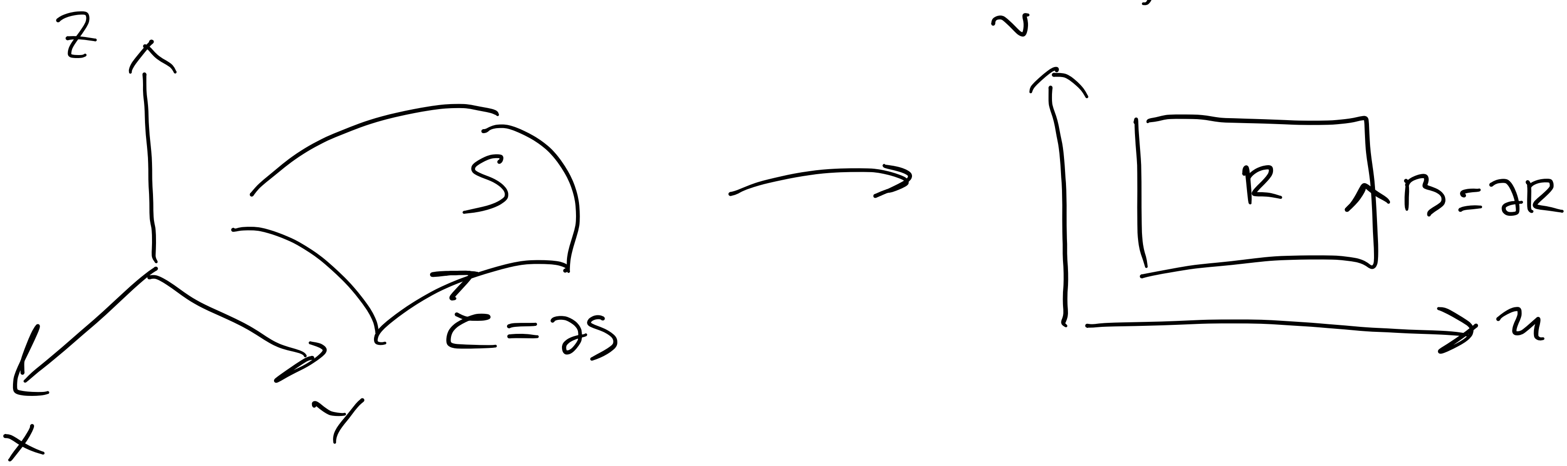
$$\mathbb{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$$



Parametisering av S: $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$

Definiera ett nytt vektorfält:

$$\mathbb{G}(u,v)$$

i uv-planet, med komponenter:

$$\begin{cases} G_1(u,v) = \mathbb{F}(\mathbf{r}(u,v)) \cdot \dfrac{\partial \mathbf{r}}{\partial u} \\ G_2(u,v) = \mathbb{F}(\mathbf{r}(u,v)) \cdot \dfrac{\partial \mathbf{r}}{\partial v} \end{cases}$$

Påstående:

$$\boxed{\operatorname{curl} \mathbb{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v}}$$

Bevis

$$HL \doteq \frac{\partial G_2}{\partial u} - \frac{\partial G_1}{\partial v} = \frac{\partial}{\partial u}\left(\mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial u}\right)$$

$$= \frac{\partial}{\partial u}\left(F_1 \frac{\partial x}{\partial v} + F_2 \frac{\partial y}{\partial v} + F_3 \frac{\partial z}{\partial v}\right)$$

$$- \frac{\partial}{\partial v}\left(F_1 \frac{\partial x}{\partial u} + F_2 \frac{\partial y}{\partial u} + F_3 \frac{\partial z}{\partial u}\right)$$

$= \{$kedjeregel & produktregel$\}$

$$= \left(\frac{\partial F_1}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z} \frac{\partial z}{\partial u}\right) \frac{\partial x}{\partial v}$$

$$+ F_1 \frac{\partial^2 x}{\partial u \partial v} + \cdots$$

$$- \Bigg[\left(\frac{\partial F_1}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z} \frac{\partial z}{\partial v}\right) \frac{\partial x}{\partial u}$$

$$+ F_1 \frac{\partial^2 x}{\partial v \partial u} + \cdots \Bigg]$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) + \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right)$$

$$+ \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right)$$

$$+ \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)$$

Nu studerar vi vänsterledet:

$$VL = \operatorname{curl} \mathbb{F} \cdot \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$$

(1) $$\operatorname{curl} \mathbb{F} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \mathbb{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \mathbb{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \mathbb{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

(2) $$\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \mathbb{i}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) - \mathbb{j}\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)$$

$$+ \mathbb{k}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)$$

$$\Rightarrow \quad VL = \operatorname{curl} \mathbb{F} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)$$

$$+ \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right)$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right)$$

$$+ \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial y}{\partial v}\frac{\partial z}{u}\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right)$$

$$+ \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)$$

$$\Rightarrow \quad VL = HL \qquad \square$$