OPTIONER OCH MATEMATIK (CTH[$TMA155$],GU[$MAM690$])

LÖSNINGAR

Skrivningsdag, skrivtid, sal: 20 maj 2006, 4 timmar, v

Inga hjälpmedel.

Varje uppgift ger maximalt 3 poäng.

1. (The one period binomial model, where $d < r < u$) Consider a call with the payoff $Y = (S(1) - K)^+$ at the termination date 1, where $S(0)e^d < K < S(0)e^u$. (a) Find the call price $\Pi_Y(0)$ at time 0. (b) Prove that $e^{-r}Y > \Pi_Y(0)$ if and only if $S(1) = S(0)e^u$.

Solution: (a) Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. We have

$$\begin{aligned}\Pi_Y(0) &= e^{-r}(q_u(se^u - K)^+ + q_d(se^d - K)^+) \\ &= e^{-r}q_u(se^u - K)\end{aligned}$$

where

$$q_u = \frac{e^r - e^d}{e^u - e^d}.$$

$ANSWER: \Pi_Y(0) = e^{-r}q_u(se^u - K)$

(b) The event

$$\left[e^{-r}Y > \Pi_Y(0)\right] = \left[(S(1) - K)^+ > q_u(se^u - K)\right]$$
$$= [S(1) > K + q_u(se^u - K)] = [S(1) > (1 - q_u)K + q_u se^u]$$
$$= [X = u] = [S(1) = S(0)e^u]$$

which proves the assertion in Problem 1(b).

2. A random variable X has the density function $f(x) = \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x \in \mathbf{R}$. Find the characteristic function $c_X(\xi) = E\left[e^{i\xi X}\right]$, $\xi \in \mathbf{R}$.

Solution. By partial integration,

$$\sqrt{2\pi}E\left[e^{i\xi X}\right] = \int_{-\infty}^{\infty} e^{i\xi x}x^2 e^{-\frac{x^2}{2}}dx$$

$$= \left[-e^{i\xi x}xe^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-i\xi e^{i\xi x}x - e^{i\xi x})e^{-\frac{x^2}{2}}dx$$

$$= \int_{-\infty}^{\infty}(i\xi e^{i\xi x}x + e^{i\xi x})e^{-\frac{x^2}{2}}dx = i\xi\int_{-\infty}^{\infty} e^{i\xi x}xe^{-\frac{x^2}{2}}dx + \sqrt{2\pi}e^{-\frac{\xi^2}{2}}$$

$$= i\xi\left\{\left[-e^{i\xi x}e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + i\xi\int_{-\infty}^{\infty} e^{i\xi x}e^{-\frac{x^2}{2}}dx\right\} + \sqrt{2\pi}e^{-\frac{\xi^2}{2}}$$

$$= (1-\xi^2)\sqrt{2\pi}e^{-\frac{\xi^2}{2}}.$$

$ANSWER: c_X(\xi) = (1-\xi^2)e^{-\frac{\xi^2}{2}}.$

3. (Black-Scholes model) Suppose $t_0 < t_* < T$ and consider a financial derivative of European type with payoff $Y =\mid S(T) - S(t_*) \mid$ at time of maturity T. Find the delta $\Delta(t)$ of the derivative at time t if

(a) $t \in \,]t_*, T[\,$. (Hint: Problem 5).

(b) $t \in \,]t_0, t_*[\,$.

(c) Finally, compute $\Delta(t_*-) - \Delta(t_*+)$.

Solution. (a) We have

$$Y = (S(T) - S(t_*))^+ + (S(t_*) - S(T))^+$$

and, hence, for all $t_* \leq t < T$,

$$\Pi_Y(t) = c(t, S(t), S(t_*), T) + p(t, S(t), S(t_*), T).$$

Thus by put-call parity,

$$\Pi_Y(t) = 2c(t, S(t), S(t_*), T) + S(t_*)e^{-r(T-t)} - S(t) \text{ if } t_* \leq t < T$$

and we get

$$\Delta(t) = 2\Phi(\frac{1}{\sigma\sqrt{T-t}}(\ln\frac{S(t)}{S(t_*)} + (r + \frac{\sigma^2}{2})(T-t))) - 1 \text{ if } t_* < t < T$$

$\leftarrow ANSWER$

(b) Since

$$c(t_*, S(t_*), S(t_*), T) = aS(t_*)$$

where

$$a = \Phi(\frac{1}{\sigma}(r + \frac{\sigma^2}{2})\sqrt{T-t_*}) - e^{-r(T-t_*)}\Phi(\frac{1}{\sigma}(r - \frac{\sigma^2}{2})\sqrt{T-t_*})$$

we get

$$\Pi_Y(t_*) = (2a + e^{-r(T-t_*)} - 1)S(t_*).$$

Thus by the dominance principle for all $t_0 < t < t_*$,

$$\Pi_Y(t) = (2a + e^{-r(T-t_*)} - 1)S(t)$$

$$= (2a + e^{-r(T-t_*)} - 1)S(t)$$

and we have

$$\Delta(t) = 2a + e^{-r(T-t_*)} - 1$$

$\leftarrow ANSWER$

(c) From the above,

$$\Delta(t_*-) - \Delta(t_*+) = e^{-r(T-t_*)}(1 - 2\Phi(\frac{1}{\sigma}(r - \frac{\sigma^2}{2})\sqrt{T-t_*}))$$

$\leftarrow ANSWER$

4. Let $W = (W(t))_{t\geq 0}$ be a standard Brownian motion. (a) Prove that $W(s) - W(t) \in N(0, \mid s-t \mid)$. (b) Suppose a is a strictly positive real number and set $X = (\frac{1}{\sqrt{a}}W(at))_{t\geq 0}$. Prove that X is a standard Brownian motion.

5. (Black-Scholes model) Suppose $\tau = T - t > 0$ and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau).$$

Prove that

$$\frac{\partial c}{\partial s}(t, s, K, T) = \Phi(d_1)\ .$$

(Hint: $c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$, where $d_2 = d_1 - \sigma\sqrt{\tau}$)