OPTIONER OCH MATEMATIK (CTH[$TMA155$],GU[$MAM690$])

LÖSNINGAR

Skrivningsdag, tid, sal: 2 sep 2006, fm, v

Inga hjälpmedel.

Skrivtid: 4 timmar

Varje uppgift ger maximalt 3 poäng.

1. (The one period binomial model, where $d < 0 < r < u$) Consider a call with the payoff $Y = (S(1) - S(0))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = s(e^u - 1)$$

and

$$h_S se^d + h_B B(0)e^r = 0.$$

From this it follows that

$$h_S s(e^u - e^d) = s(e^u - 1)$$

and

$$h_S = \frac{e^u - 1}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)} h_S se^{d-r} = \frac{se^{d-r}}{B(0)} \frac{1 - e^u}{e^u - e^d}.$$

$ANSWER:$ $\frac{e^u-1}{e^u-e^d}$ units of the stock and $\frac{se^{d-r}}{B(0)}\frac{1-e^u}{e^u-e^d}$ units of the bond.

2. (Black-Scholes model) Suppose $0 < t < T$ and consider a financial derivative of European type with payoff $Y = (S(T) - S(0))^2/S(T)$ at time of

maturity T. Find the price $\Pi_Y(t)$ and the delta $\Delta(t)$ of the derivative at time t.

Solution. We have

$$Y = S(T) - 2S(0) + S(0)^2 S(T)^{-1}.$$

Here

$$\Pi_{S(T)}(t) = S(t)$$

and

$$\Pi_{S(0)}(t) = S(0)e^{-r\tau}$$

where $\tau = T - t$. Moreover,

$$\Pi_{S(T)^{-1}}(t) = e^{-r\tau} \int_{-\infty}^{\infty} (S(t)e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x})^{-1} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= S(t)^{-1} e^{-r\tau} e^{-(r-\frac{\sigma^2}{2})\tau} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} - \sigma\sqrt{\tau}x} \frac{dx}{\sqrt{2\pi}}$$

$$= S(t)^{-1} e^{(\sigma^2 - 2r)\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} = S(t)^{-1} e^{(\sigma^2-2r)\tau}.$$

Hence

$$\Pi_Y(t) = S(t) - 2S(0)e^{-r\tau} + S(0)^2 e^{(\sigma^2-2r)\tau} S(t)^{-1} \leftarrow ANSWER$$

Now, if $\Pi_Y(t) = v(t, S(t))$,

$$\Delta(t) = \frac{\partial v}{\partial s}(t, S(t)) = 1 - S(0)^2 e^{(\sigma^2-2r)\tau} S(t)^{-2} ANSWER$$

3. Suppose X is a non-negative random variable with probability density f and such that $0 < E[X^2] < \infty$. Let $\mu = E[X]$ and suppose $\alpha \in [0,1]$.
(a) Prove that

$$\int_{\alpha\mu}^{\infty} x f(x) dx \geq (1-\alpha)\mu.$$

(b) Prove that

$$\int_{\alpha\mu}^{\infty} f(x)dx \geq (1-\alpha)^2 \frac{(E[X])^2}{E[X^2]}.$$

Solution. (a) We have

$$0 \leq \mu = \int_0^{\alpha\mu} xf(x)dx + \int_{\alpha\mu}^{\infty} xf(x)dx$$

$$\leq \int_0^{\alpha\mu} \alpha\mu f(x)dx + \int_{\alpha\mu}^{\infty} xf(x)dx \leq \int_0^{\infty} \alpha\mu f(x)dx + \int_{\alpha\mu}^{\infty} xf(x)dx$$

$$= \alpha\mu + \int_{\alpha\mu}^{\infty} xf(x)dx$$

and, consequently,

$$\int_{\alpha\mu}^{\infty} xf(x)dx \geq (1-\alpha)\mu.$$

(b) We have

$$\int_{\alpha\mu}^{\infty} xf(x)dx = \int_0^{\infty} x\sqrt{f(x)} 1_{[\alpha\mu,\infty[}(x)\sqrt{f(x)}dx$$

and the Cauchy-Schwarz inequality yields

$$\int_{\alpha\mu}^{\infty} xf(x)dx \leq (\int_0^{\infty} x^2 f(x)dx)^{\frac{1}{2}} (\int_0^{\infty} 1_{[\alpha\mu,\infty[}(x)f(x)dx)^{\frac{1}{2}}$$

$$= (\int_0^{\infty} x^2 f(x)dx)^{\frac{1}{2}} (\int_{\alpha\mu}^{\infty} f(x)dx)^{\frac{1}{2}}$$

and, hence,

$$\int_{\alpha\mu}^{\infty} f(x)dx \geq \frac{(\int_{\alpha\mu}^{\infty} xf(x)dx)^2}{\int_0^{\infty} x^2 f(x)dx}$$

$$\geq \frac{(1-\alpha)^2\mu^2}{\int_{-\infty}^{\infty} x^2 f(x)dx} = (1-\alpha)^2 \frac{(E[X])^2}{E[X^2]}.$$

4. (Dominance Principle) Show that the map

$$K \to c(t, S(t), K, T), \ \ K > 0$$

is convex.

5. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. such that $P\,[X_1 = 1] = P\,[X_1 = -1] = \frac{1}{2}$ and set

$$Y_n = \frac{1}{\sqrt{n}}(X_1 + ... + X_n),\ \ n \in \mathbf{N}_+.$$

Prove that $Y_n \to G$, where $G \in N(0,1)$.