OPTIONS AND MATHEMATICS
(CTH[$TMA155$], GU[$MAM690$])

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No aids.
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Each problem is worth 3 points.

Solutions

1. (The one period binomial model, where $d < 0 < r < u$) Suppose

$$S(0)e^d < K < S(0)e^u$$

and consider a put of European type with the payoff $Y = (K - S(1))^+$ at the termination date 1. Find the replicating strategy of the derivative at time 0.

Solution: Let $S(0) = s$ and $S(1) = se^X$, where $X = u$ or d. If (h_S, h_B) denotes the replicating strategy at time 0 we have

$$h_S se^u + h_B B(0)e^r = 0$$

and

$$h_S se^d + h_B B(0)e^r = K - se^d.$$

From this it follows that

$$h_S s(e^u - e^d) = se^d - K$$

and

$$h_S = \frac{1}{s}\frac{se^d - K}{e^u - e^d}.$$

Moreover, we get

$$h_B = -\frac{1}{B(0)}h_S se^{u-r} = \frac{e^{u-r}}{B(0)}\frac{K - se^d}{e^u - e^d}.$$

$ANSWER: \frac{1}{S(0)}\frac{se^d-K}{e^u-e^d}$ units of the stock and $\frac{e^{u-r}}{B(0)}\frac{K-se^d}{e^u-e^d}$ units of the bond.

2. (Black-Scholes model) Suppose $0 < t < T$ and consider a financial derivative of European type with payoff

$$Y = \begin{cases} 1 \text{ if } S(T) > K \\ 0 \text{ if } S(T) \leq K \end{cases}$$

at time of maturity T. Find the price $\Pi_Y(t)$ and the delta $\Delta(t)$ of the derivative at time t. For which value of the stock price $S(t)$ is $\Delta(t)$ maximal?

Solution. We have

$$Y = g(S(T))$$

where

$$g(x) = \begin{cases} 1 & \text{if } x > K \\ 0 & \text{if } x \leq K. \end{cases}$$

Thus, if $s = S(T)$ and $\tau = T - t$,

$$\Pi_Y(t) = e^{-r\tau} \int_{-\infty}^{\infty} g(se^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}x}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{-r\tau} \int_{-d_2}^{\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = e^{-r\tau}\Phi(d_2)$$

where

$$d_2 = \frac{1}{\sigma\sqrt{\tau}}(\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})\tau).$$

From this we get

$$\Delta(t) = \frac{\partial}{\partial s}\Pi_Y(t) = \frac{e^{-r\tau}}{s\sigma\sqrt{2\pi\tau}} e^{-\frac{d_2^2}{2}}$$

and

$$\frac{\partial}{\partial s}\Delta(t) = -\frac{1}{s^2}\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{d_2^2}{2}} \left(1 + \frac{1}{\sigma^2\tau}(\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})\tau)\right).$$

Thus $\frac{\partial}{\partial s}\Delta(t) = 0$ if $s = s_*$, where

$$s_* = Ke^{-(r+\frac{\sigma^2}{2})\tau}.$$

Moreover, $\frac{\partial}{\partial s}\Delta(t) > 0$ if $s < s_*$ and $\frac{\partial}{\partial s}\Delta(t) < 0$ if $s > s_*$ and it follows that the delta of the option has a maximum for $S(t) = s_*$.

3. Set $X(t) = W(t) - tW(1)$ and $Y(t) = X(1-t)$ if $0 \leq t \leq 1$. Prove that the processes $(X(t))_{0 \leq t \leq 1}$ and $(Y(t))_{0 \leq t \leq 1}$ are equivalent in distribution.

Solution. Given $t_1, ..., t_n \in [0,1]$ an arbitrary linear combination of $X(t_1), ..., X(t_n)$ is a linear combination of $W(t_1), ..., W(t_n), W(1)$ and, hence a centred Gaussian random variable. In a similar way a linear combination of $Y(t_1), ..., Y(t_n)$ is a centred Gaussian random variable. Therefore it only remains to prove that the processes $(X(t))_{0 \leq t \leq 1}$ and $(Y(t))_{0 \leq t \leq 1}$ have the same covariance. To this end let $0 \leq s \leq t \leq 1$. Then

$$E\left[X(s)X(t)\right] = E\left[(W(s) - sW(1))(W(t) - tW(1)\right]$$

$$= E\left[W(s)W(t)\right] - tE\left[W(s)W(1)\right] - sE\left[W(1)W(t)\right] + stE\left[(W^2(1)\right]$$

$$= s - st - st + st = s - st$$

and

$$E\left[Y(s)Y(t)\right] = E\left[X(1-t)X(1-s)\right] = (1-t) - (1-t)(1-s) = s - st.$$

Thus $E\left[X(s)X(t)\right] = E\left[Y(s)Y(t)\right] = \min(s,t) - st$ for all $0 \leq s, t \leq 1$ and it follows that the processes $(X(t))_{0 \leq t \leq 1}$ and $(Y(t))_{0 \leq t \leq 1}$ are equivalent in distribution.

4. Suppose $a > 0$. Prove the Markov inequality

$$P\left[\mid X \mid \geq a\right] \leq \frac{1}{a} E\left[\mid X \mid\right].$$

5. (Black-Scholes model) Suppose $t < T$ and $\tau = T - t$. Prove that

$$c(t, s, K, T) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

and

$$p(t, s, K, T) = Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1)$$

where

$$d_1 = \frac{\ln \frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and

$$d_2 = \frac{\ln \frac{s}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$