OPTIONER OCH MATEMATIK (CTH[$TMA155$],GU[$MAM690$])

ASSIGNMENTS 2006
Must be handed in at the latest Thursday, April 27 at 10^{45}

Problem 1 Suppose $t < T$ and $c(t, S(t), K, T) = 3.14$. The price of a European derivative at time t is N units of currency and it pays at the maturity date T the amount

$$N + \alpha N(S(T) - K)^{+}.$$

If $N \neq 0$, show that

$$\alpha = (1 - e^{-r(T-t)})/3.14.$$

Problem 2 Below θ_i, $i = 0, 1, ..., n$, are positive real numbers such that $\Sigma_0^n \theta_i = 1$.

(a) Suppose $f : I \to \mathbf{R}$ is convex and $x_0, ..., x_n \in I$, where I is a subinterval of $\mathbf{R}$. Show that

$$f(\Sigma_0^n \theta_i x_i) \leq \Sigma_0^n \theta_i f(x_i).$$

(Hint: Use induction on n.)

(b) Suppose $a_i > 0$, $i = 0, 1, ..., n$. Show that

$$\Pi_0^n a_i^{\theta_i} \leq \Sigma_0^n \theta_i a_i.$$

(Hint: If $f :]0, \infty[\to \mathbf{R}$ is differentiable and f' increasing, then f is convex.)

(c) Suppose $K > 0$ and $t \leq t_0 \leq ...t_n \leq T$. Two European derivatives with maturity date T have the payoffs

$$Y_1 = (\Pi_0^n S(t_i)^{\theta_i} - K)^{+}$$

and

$$Y_2 = (\Sigma_0^n \theta_i S(t_i) - K)^{+},$$

respectively. Show that

$$\Pi_{Y_1}(t) \leq \Pi_{Y_2}(t) \leq c(t, S(t), K, T).$$

Problem 3 Suppose X is is a random variable such that

$$P[X=-1]=P[X=1]=\frac{1}{2}.$$

Find all $\lambda \in \mathbf{R}$ such that

$$E\left[(a+\lambda bX)^4\right] \leq \left(E\left[(a+bX)^2\right]\right)^2$$

for all a, $b \in \mathbf{R}$. (Hint: $X = X^3$ and $X^2 = X^4 = 1$.)

Problem 4 Consider the equation

$$\frac{\partial u}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+a\frac{\partial u}{\partial x}+bu=0$$

where $a, b \in \mathbf{R}$ and $\sigma > 0$. Find $\alpha, \beta \in \mathbf{R}$ so that the substitution

$$u(t,x)=e^{\alpha t+\beta x}v(t,x)$$

leads to the simpler equation

$$\frac{\partial v}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}=0\ .$$

Problem 5 (a) Suppose $G \in N(0,1)$. Compute $E\left[G^4\right]$.

(b) In the compendium 'Introduction to the Black-Scholes Theory', Chapter 4, p 12, you read the following:

Next we will discuss some statistical estimates of the parameters α and σ in the geometric Brownian motion model. To this end fix a period of time from 0 to T and choose a natural number n. Set $h = T/n$, $t_i = ih$, $i = 1, ..., n$, and

$$X_i=\ln\frac{S(t_i)}{S(t_{i-1})}=\alpha h+\sigma\sqrt{h}G_i,$$

for $i = 1, ..., n,$ *where* $G_1, ..., G_n \in N(0,1)$ *are independent. Furthermore, define*

$$\hat{\alpha} = \frac{1}{T}\Sigma_{i=1}^{n} X_i$$

and

$$\hat{\sigma}^2 = \frac{1}{T}\Sigma_{i=1}^{n} X_i^2.$$

It is readily seen that

$$\begin{cases} E\left[\hat{\alpha}\right] = \alpha \\ \mathrm{Var}(\hat{\alpha}) = \frac{\sigma^2}{T} \end{cases}$$

and after some calculations the following formulas are obtained, viz.

$$\begin{cases} E\left[\hat{\sigma}^2\right] = \sigma^2 + \frac{\alpha^2 T}{n} \\ \mathrm{Var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n} + \frac{4\alpha^2 T\sigma^2}{n^2}. \end{cases}$$

Give a detailed derivation of the last two equations.