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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination 17 November 2008

1a) $A = LL^T$ by Cholesky, $Ly = x$ by forward substitution and then the scalar product $y^T y$, since $y^T y = x^T L^{-T} L^{-1} x = x^T A^{-1} x$.

1b) Let $\begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} = QR$ be a QR-factorization. Then

$$A^T A + \alpha I = \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix}^T \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} = (QR)^T (QR) = R^T Q^T Q R = R^T R.$$

2) See text book or lecture notes.

3) Let $B^T = [Q \quad \tilde{Q}] \begin{bmatrix} R \\ 0 \end{bmatrix}$ be the full QR-factorization of B^T. Then $Bx = d \Leftrightarrow [R^T \quad 0] \begin{bmatrix} Q^T \\ \tilde{Q}^T \end{bmatrix} x = d$. By the orthogonal variabel transformation $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Q^T x \\ \tilde{Q}^T x \end{bmatrix}$, we then get $x = Qy_1 + \tilde{Q} y_2$, where y_2 is free and $y_1 = R^{-T} d$ (formally).
Now, the over-determined system $Ax = b \Leftrightarrow AQR^{-T} d + A\tilde{Q} y_2 - b = 0$. In least squares sense, this problem is solved for y_2 by the compact QR-factorization of $A\tilde{Q} = Q_1 R_1$. The solution is then formally: $y_2 = R_1^{-1} Q_1^T (b - AQR^{-T} d)$.

4a) $R(1,2,\theta) = \begin{bmatrix} c & -s & & & \\ s & c & & & \\ & & 1 & & \\ & & & \cdots & \\ & & & & 1 \end{bmatrix}$. The eigenvalues are $n-2$ ones and the two eigenvalues of $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$. The latter are $\lambda = c \pm is$ with corresponding right and left eigenvectors $x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mp i \end{bmatrix}$ and $y^* = \frac{1}{\sqrt{2}} [1 \quad \mp i]$. The condition number becomes $\kappa(\lambda) = \frac{1}{|y^* x|} = 1/1 = 1$.

4b) $|\delta \lambda| \le \frac{1}{|y^* x|} \|\delta A\|_2 + \mathcal{O}(\|\delta A\|_2^2)$.

5 The Householder matrix is computed by $\tilde{u} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} \Rightarrow$

$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow H = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

The similar transformation then reads $HA = \begin{bmatrix} 3 & 1 & -1 \\ 4 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and $HAH = \begin{bmatrix} 3 & -1 & 1 \\ 4 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$.

6a)
The algebraic multiplicity of the eigenvalue λ_i is the multiplicity of the root λ_i of the characteristic equation $det(A - \lambda_i I) = 0$.
The geometric multiplicity of λ_i is the dimension of the eigenspace $L(\lambda_i)$, that is $n - rank(A - \lambda_i I)$ for an $n \times n$ matrix.
An eigenvalue is defect if the geometric multiplicity is smaller than the algebraic. A typical example is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, i.e. A is a so called Jordan-block.

6b) $A = V^T AV$ where V is orthogonal and T is block upper triangular with diagonal blocks of size 1×1 or 2×2.

7) Use $R(1, 3, \theta)$ to zero-out the (3,1) and (1,3) elements:

$$R^T AR = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix} = \begin{bmatrix} 2(c+s)^2 & c+s & 2(c^2-s^2) \\ c+s & 2 & -s+c \\ 2(c^2-s^2) & -s+c & 2(s-c)^2 \end{bmatrix}.$$

Now, take $s = c = \frac{1}{\sqrt{2}}$ to get $R^T AR = \begin{bmatrix} 4 & \frac{2}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{2}} & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

8) See text book or lecture notes.