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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination 2 December 2009

1a) Small errors in data (the input) result in small errors in the answer to the problem (the output).
1b) The algorithm gives the exact solution to a slightly pertubed problem.
1c) $Ax = b, \quad A(x + \delta x) = b + \delta b \Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|} = \kappa(A)\frac{\|\delta b\|}{\|b\|}$

2) See text book or lecture notes.

3a) $A = LDL^T$, L lower triangular, D block diagonal with 1x1 or 2x2 diagonal blocks.
3b) Let $B = LL^T$ be the Cholesky factorization with L nonsingular. Then $Ax = \lambda Bx \Leftrightarrow AL^{-T}L^Tx = \lambda LL^Tx \Leftrightarrow L^{-1}AL^{-T}(L^Tx) = \lambda(L^Tx) \Leftrightarrow \tilde{A}y = \lambda y$, where $\tilde{A} = L^{-1}AL^{-T}$ is symmetric and $L^Tx = y$.

4) For $H = I - 2uu^T$ first calculate $\hat{u} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ and normalize to $u = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$. The Householder reflection becomes $H = I - 2uu^T$ and $HA = A - 2uu^TA =$ ”column by column” $= \begin{bmatrix} 0 & -1 & 1 \\ 5 & 4 & 0 \\ 0 & -2 & 0 \end{bmatrix}$. Then $(HA)H = HA - 2(HA)uu^T =$ ”row by row”
$= \begin{bmatrix} 0 & -1/5 & -7/5 \\ 5 & 16/5 & 12/5 \\ 0 & -8/5 & -6/5 \end{bmatrix}$

5a) $[A\ B]$ nonsingular $\Rightarrow$ A and B have full rank $\Rightarrow$ $A^+ = (A^TA)^{-1}A^T,\ \ B^+ = (B^TB)^{-1}B^T$. Further $A^TB = O \Rightarrow A^+B = (A^TA)^{-1}A^TB = 0$ and $A^TB = 0 \Rightarrow B^TA = O \Rightarrow B^+A = (B^TB)^{-1}B^TA = O$. Finally, $\begin{bmatrix} A^+ \\ B^+ \end{bmatrix} [A\ B] = \begin{bmatrix} A^+A & A^+B \\ B^+A & B^+B \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I.$
5b) SVD and definition of Moore Penrose pseudoinverse give $A = U\Sigma V^T$, $A^+ = V\Sigma^+U^T$ and $A^+A = V\Sigma^+\Sigma V^T$. Here, $\Sigma^+\Sigma$ is quasidiagonal with ones and zeros in the diagonal. It follows that the singular values of A^+A are ones and zeros. Finally $\|A^+A\|_2$ is the maximum singular value of A^+A and its value is 1.

6a) $Q^*AQ = T$, where Q is unitary and T is upper triangular.
6b) $(A+E)\bar{x} = A\bar{x} + E\bar{x} = \bar{\lambda}\bar{x} + r + E\bar{x}$, which equals $\bar{\lambda}\bar{x}$ if $E\bar{x} = -r$ and then $(\bar{\lambda}, \bar{x})$ is an eigenpair of $A+E$. This equation holds for $E = -\frac{r\bar{x}^T}{\bar{x}^T\bar{x}}$ and then $\|E\|_2 = \frac{\|r\|_2}{\|\bar{x}\|_2}$ since $\|r\bar{x}^T\|_2 = max_{y\neq 0}\frac{\|r\bar{x}^Ty\|_2}{\|y\|_2} = max_{y\neq 0}\frac{|\bar{x}^Ty|\|r\|_2}{\|y\|_2} = \|\bar{x}\|_2\|r\|_2$, where the maximum is attained for $y = \bar{x}$.

7) See text book or lecture notes.

8) Use $R(1,3,\theta)$ to zero-out the (3,1) and (1,3) elements:

$$R^TAR = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix} = \begin{bmatrix} 2(c+s)^2 & s & 2(c^2-s^2) \\ s & 1 & c \\ 2(c^2-s^2) & c & 2(s-c)^2 \end{bmatrix}.$$

Now, take $s = c = \frac{1}{\sqrt{2}}$ to get $R^TAR = \begin{bmatrix} 4 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$.