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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination 17 December 2010

1a) Symmetric: $A^T = A$. Indefinite: $x^T Ax > 0$ for some x and $y^T Ay < 0$ for some y.
b) $A = LL^T$ by Cholesky, $Ly = x$ by forward substitution and then the scalar product $y^T y$, since $y^T y = x^T L^{-T} L^{-1} x = x^T A^{-1} x$.

2) See text book or lecture notes.

3) (1): $\|A\|_2^2 = \max_x \frac{x^T A^T Ax}{x^T x}$ = (by diagonalization) $= \max_x \frac{x^T Y^T DYx}{x^T x}$ = (since $y = Yx$ and Y is orthogonal) $= \max_y \frac{y^T Dy}{y^T y} = \lambda_{max}(A^T A)$.
$A = U\Sigma V^T \Rightarrow A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$ is a diagonalization of $A^T A$, so
(2): λ eigenvalue of $A^T A$ if and only if $\sqrt{\lambda}$ singular value of A.
(1) and (2) give: $\|A\|_2 = \sigma_1$, the largest singular value of A.
$A - A_k = \sum_{i=1}^n \sigma_i u_i v_i^T - \sum_{i=1}^k \sigma_i u_i v_i^T = \sum_{i=k+1}^n \sigma_i u_i v_i^T = \hat{U}\hat{\Sigma}\hat{V}^T$ and this matrix has largest singular value σ_{k+1} that is $\|A - A_k\|_2 = \sigma_{k+1}$.
4a) $H = I - 2uu^T$, $H^T = (I - 2uu^T)^T = I - 2uu^T = H$, $H^T H = (I + uu^T)(I + uu^T) = I - 2uu^T - 2uu^T + 4uu^T = I$.
4b) $Hx = x - uu^T x = x - (u^T x)u$ with $\|Hx\|_2 = \|x\|_2$, since H is orthogonal, so H is a reflection in a plane orthogonal to the vector u.

4c) Find the Householder reflection $H = I - 2uu^T$. Let $\hat{u} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \\ -3 \end{bmatrix} \Rightarrow$

$u = \frac{1}{5\sqrt{2}} \begin{bmatrix} 5 \\ -4 \\ 0 \\ -3 \end{bmatrix}$. The second column then becomes $H \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - 2(-25)\frac{1}{25\cdot 2} \begin{bmatrix} 5 \\ -4 \\ 0 \\ -3 \end{bmatrix} =$
$\begin{bmatrix} 4 \\ -2 \\ 3 \\ 1 \end{bmatrix}$ and the first step is completed in $A^{(2)} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$.

5 $G(\theta) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$, where $c = \cos(\theta)$ and $s = \sin(\theta)$. The eigenvalues are $c \pm is$ with right eigenvectors $x = \pm\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$ and left eigenvectors $\bar{y} = \pm\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$. The condition number is then $\kappa_\lambda = \frac{1}{|y^* x|}$ and $|y^* x| = \frac{1}{2}[1 \quad -i] \begin{bmatrix} 1 \\ i \end{bmatrix}$ so $\kappa_\lambda = 1$.

6a) $\|Qx\|_2^2 = x^T Q^T Q x = x^T x = \|x\|_2^2$, since Q is orthogonal i.e. $Q^T Q = I$.
We conclude that $\|Qx\|_2 = \|x\|_2$.
6b) We have from the QR-factorization: $A = QR$, where $Q = [Q_1 \; Q_2]$ and the compact QR-factorization is $A = Q_1 R$ and $Q_1^T A = R$. Since the 2-norm is invariant under orthogonal transformations, by 6a), we get
$\|Ax - b\|_2^2 = \|Q^T(Ax - b)\|_2^2 = \| \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} (Ax - b)\|_2^2 = \| \begin{bmatrix} R \\ O \end{bmatrix} x - \begin{bmatrix} Q_1^T b \\ Q_2^T b \end{bmatrix} \|_2^2 =$
$= \|Rx - Q_1^T b\|_2^2 + \|Q_2^T b\|_2^2$ and this norm is minimized for $Rx = Q_1^T b$, an upper triangular system to be solved.

7) See text book or lecture notes.

8a) Use $R(1, 3, \theta)$ to zero-out the (3,1) and (1,3) elements:

$$R^T A R = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix} = \begin{bmatrix} 2(c+s)^2 & c+s & 2(c^2 - s^2) \\ c+s & 1 & -s+c \\ 2(c^2 - s^2) & -s+c & 2(s-c)^2 \end{bmatrix}.$$

Now, take $s = c = \frac{1}{\sqrt{2}}$ to get $R^T A R = \begin{bmatrix} 4 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

8b See text book or lecture notes.