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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination 23 October 2009

1) $$\left.\begin{array}{r} A^T A(x+\delta x) = A^T(b+\delta b) \\ A^T Ax = A^T b \end{array}\right\} \Rightarrow A^T A\delta x = A^T \delta b \Rightarrow \delta x = (A^T A)^{-1} A^T \delta b = A^+ \delta b$$
$\Rightarrow \|\delta x\|_2 \le \|A^+\|_2 \|\delta b\|_2 \Rightarrow \frac{\|\delta x\|_2}{\|x\|_2} \le \|A^+\|_2 \frac{\|\delta b\|_2}{\|x\|_2} = \|A^+\|_2 \|A\|_2 \frac{\|b\|_2 \|\delta b\|_2}{\|A\|_2 \|b\|_2 \|x\|_2} = \kappa(A) \frac{\|b\|_2 \|\delta b\|_2}{\|A\|_2 \|b\|_2 \|x\|_2}$.
By the inequality $\|Ax\|_2 \le \|A\|_2 \|x\|_2$ and the fact that $\cos\theta = \frac{\|Ax\|_2}{\|b\|_2}$ we now get
$\frac{\|\delta x\|_2}{\|x\|_2} \le \kappa(A) \frac{\|b\|_2 \|\delta b\|_2}{\|Ax\|_2 \|b\|_2} = \kappa(A) \frac{1}{\cos\theta} \frac{\|\delta b\|_2}{\|b\|_2}$.

2) See text book or lecture notes.

3a) If x_0 is a solution to $Ax = b$, then all $x = x_0 + x_h$, where $x_h \in N(A)$, are solutions, since $Ax = Ax_0 + Ax_h = Ax_0 = b$.
$\|x\|_2$ is minimized if $x \in N(A)^\perp = R(A^T)$ i.e. for $x = A^T y$ for some $y \in R^m$ and then $b = Ax = AA^T y$ i.e. the problem is solved by
1) $AA^T y = b$, 2) $x = A^T y$.
3b) $(AA^T)^T = (A^T)^T A^T = AA^T$ so AA^T is symmetric.
$x^T AA^T x = (A^T x)^T (A^T x) \ge 0$ and $(A^T x)^T (A^T x) = 0 \Leftrightarrow A^T x = 0 \Leftrightarrow x = 0$, since A^T has full rank, so AA^T is positive definite.

4) Use a Givens rotation $R(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$ to zero-out the (3,1) element:
$R(2,3,\theta)A = \begin{bmatrix} 0 & -1 & 1 \\ 4c-3s & 2c-4s & 0 \\ 4s+3c & 2s+4c & 0 \end{bmatrix}$. By $s = -3/5$, $c = 4/5$ we get the desired
$R(2,3,\theta)A = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 4 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and then
$$R(2,3,\theta)AR(2,3,\theta)^T = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 4 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 0 & -1/5 & -7/5 \\ 5 & 16/5 & -12/5 \\ 0 & 8/5 & -6/5 \end{bmatrix}.$$

5a) $G = U\Sigma V^T$ where $U \in R^{m\times m}$ and $V \in R^{n\times n}$ are orthogonal and $\Sigma \in R^{m\times n}$ is quasi-diagonal with the singular values of G in decreasing order $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$, $\sigma_{r+1} = \sigma_{r+2} = \sigma_n = 0$.

5b) $G = U\Sigma V^T \Rightarrow H = \begin{bmatrix} O & V\Sigma U^T \\ U\Sigma V^T & O \end{bmatrix}$. Now, $W = \frac{1}{\sqrt{2}}\begin{bmatrix} V & V \\ U & -U \end{bmatrix}$ is orthogonal since $W^TW = \frac{1}{2}\begin{bmatrix} VV^T + VV^T & O \\ O & UU^T + UU^T \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I$.

Further $W\begin{bmatrix} \Sigma & O \\ O & \Sigma \end{bmatrix}W^T = \begin{bmatrix} O & V\Sigma U^T \\ U\Sigma V^T & O \end{bmatrix} = H$, so $D = \begin{bmatrix} \Sigma & O \\ O & \Sigma \end{bmatrix}$ contains the eigenvalues and W contains the eigenvectors as columns, by the spectral theorem.

6a For H first calculate $\hat{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ and normalize to $u = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$.

The Householder reflection becomes $H = I - 2uu^T$ and $HA = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

For Z we calculate $\hat{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and normalize to $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

The Householder reflection becomes $Z = I - 2uu^T$ and $HAZ = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

7) See text book or lecture notes.

8a) Congruent transformation: $A \to X^TAX$ with X nonsingular. The inertia is preserved, $inertia(A) = (\nu,\ \zeta,\ \pi)$ where ν, ζ and π is the number of negative, zero and positive eigenvalues, respectively.

8b) The eigenvalues of $A - \sigma I$ are $\lambda_i - \sigma$ for eigenvalues λ_i of A and $\lambda_i,\ i = 1, 2, 3, 4$ are integers (by the hint). By a) $A - \sigma I$ and D have the same inertia in particular the same π. From the table we see that A has 4 eigenvalues larger than 0, 3 eigenvalues larger than 1, 2 eigenvalues larger that 2 etc. We conclude that the eigenvalues are 1, 2, 4 and 5.

8c) By Gerschgorin: $\lambda \in \{|z-2| \leq 2\} \cup \{|z-3| \leq 2\} \cup \{|z-4| \leq 2\} \cup \{|z-3| \leq 2\} \Rightarrow \lambda \in [0,\ 6]$. The eigenvalues are real since A is symmetric.