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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination October 22, 2010

1a) $\left\{ \begin{array}{l} A(x+\delta x) = b + \delta b \\ Ax = b \end{array} \right\} \Leftrightarrow A\delta x = \delta b \Leftrightarrow \delta x = A^{-1}\delta b \Rightarrow$
$\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\|\delta b\| \Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \|A^{-1}\|\frac{\|\delta b\|}{\|x\|} = \|A^{-1}\|\|A\|\frac{\|\delta b\|}{\|A\|\|x\|} =$
$= \kappa(A)\frac{\|\delta b\|}{\|A\|\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}$, where the last inequality comes from $\|b\| = \|Ax\| \leq \|A\|\|x\|$.
1b) $A = LU,\ A^T w = u \Leftrightarrow \left\{ \begin{array}{l} U^T v = u \\ L^T w = v \end{array} \right.$
Take $u = \pm 1$ with signs to make $|v_i|$ maximal, then w is realistic since we reveal the ill-conditioning of A, inherited in U.
Finally, $\frac{1}{\kappa(A)} = \frac{1}{\|A\|\|A^{-1}\|} \leq \frac{\|w\|}{\|A\|\|x\|}$,
where the last inequality comes from $\|x\| = \|A^{-1}w\| \leq \|A^{-1}\|\|w\|$.

2a) See text book or lecture notes; $q \approx \sqrt{M/3}$.
2b) "Matrix times matrix" and "solving triangular systems with many right hand sides".

3) We have from the SVD: $A = U\Sigma V^T \Leftrightarrow U^T A = \Sigma V^T$. Let $z = V^T x$ be a transformation and let $rank(A) = r$. Since the 2-norm is invariant under orthogonal transformations we get: $\|Ax - b\|_2^2 = \|U^T(Ax-b)\|_2^2 = \|\Sigma V^T x - U^T b\|_2^2 = \|\Sigma z - U^T b\|_2^2 =$
$\| \begin{bmatrix} \Sigma_r z_1 \\ O \end{bmatrix} - \begin{bmatrix} U_1^T b \\ U_2^T b \end{bmatrix} \|_2^2 = \|\Sigma_r z_1 - U_1^T b\|_2^2 + \|U_2^T b\|_2^2$ i.e. $\|Ax - b\|_2^2$ is minimized for $z_1 = \Sigma_r^{-1} U_1^T b$. For the minimum norm note that $x = Vz$ and $\|x\|_2^2 = \|z\|_2^2 = \|z_1\|_2^2 + \|z_2\|_2^2$ is minimized for $z_2 = O$ and then $x = Vz = V_1 z_1 + V_2 z_2 = V_1 z_1 = V_1 \Sigma_r^{-1} U_1^T b$.

4a) For $H = I - 2uu^T$ first calculate $\hat{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and normalize to $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The Householder reflection becomes $H = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
Then $HAH = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$.

4b) Use a Givens rotation $R(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$ to zero-out the (3,1) element:

$R(2,3,\theta)A = \begin{bmatrix} 2 & 0 & 1 \\ s & 3c-2s & 2c+4s \\ c & -3s+2c & -2s+4c \end{bmatrix}$. By $s=1,\ c=0$ we get the desired $R(2,3,\theta)A =$
$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 4 \\ 0 & -3 & -2 \end{bmatrix}$ and then

$$R(2,3,\theta)AR(2,3,\theta)^T = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 4 \\ 0 & -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

4c) We apply spectral slicing on the tridiagonal matrix from b): $B = HAH = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$,
i.e. we want to find a factorization $B - \sigma I = LDL^T$ with $\sigma = 1$, so we should identify the elements in L and D from:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ 0 & l_2 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_1 & 0 \\ 0 & 1 & l_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d_1 & l_1 d_1 & 0 \\ l_1 d_1 & d_2 + l_1^2 d_1 & l_2 d_2 \\ 0 & l_2 d_2 & d_3 + l_2^2 d_2 \end{bmatrix}.$$

We find $d_1 = 1,\ \ l_1 = 1,\ \ d_2 = 2,\ \ l_2 = 1$ and $d_3 = 0$. The eigenvalues of D are ≥ 0 so the eigenvalues of A are ≥ 1. One eigenvalue of D is 0 so one eigenvalue of A is 1.

5a) Let $x \in R(X)$ i.e. $x = Xz$ for some z. Then $Ax = AXz = XBz \in R(X)$ so X is right invariant subspace.
5b) Let λ be an eigenvalue of B. Then $By = \lambda y$ for some eigenvektor y and then $XBy = \lambda Xy \Rightarrow AXy = \lambda Xy$ so λ is also an eigenvalue of A with eigenvector Xy.
5c) Assume X_1 contains some eigenvectors of A as columns. Then X_1 is a right invariant subspace $AX_1 = X_1 D$, where D is diagonal with corresponding eigenvalues. Let now $X = [X_1\ X_2]$ be non-singular. Then $X^{-1}AX = X^{-1}[AX_1\ AX_2] = [X^{-1}X_1 D\ \ X^{-1}AX_2] =$
$\begin{bmatrix} D & \tilde{A_{12}} \\ O & \tilde{A_{22}} \end{bmatrix}$, so the rest of the eigenvalues of A are the eigenvalues of the smaller matrix $\tilde{A_{22}}$.

6) See text book or lecture notes.

7) See text book or lecture notes.