Department of
Mathematics
GÃűteborg

Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination December 9, 2011

1a) $\left\{ \begin{array}{l} A(x+\delta x) = b+\delta b \\ Ax = b \end{array} \right\} \Leftrightarrow A\delta x = \delta b \Leftrightarrow \delta x = A^{-1}\delta b \Rightarrow$
$\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\|\delta b\| \Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \|A^{-1}\|\frac{\|\delta b\|}{\|x\|} = \|A^{-1}\|\|A\|\frac{\|\delta b\|}{\|A\|\|x\|} =$
$= \kappa(A)\frac{\|\delta b\|}{\|A\|\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}$, where the last inequality comes from $\|b\| = \|Ax\| \leq \|A\|\|x\|$.
1b) $\kappa(\lambda) = \frac{1}{|y^* x|}$, where x and y are normed left and right eigenvectors corresponding to λ.
1c) An eigenvalue is defective if the geometric multiplicity is smaller than the algebraic.
A typical example is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, i.e. A is a so called Jordan-block.

2. See text book or lecture notes.

3) We have from the full QR-factorization: $A = [Q_1\ Q_2]\begin{bmatrix} R \\ O \end{bmatrix}$, which also can be written $\begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} A = \begin{bmatrix} R \\ O \end{bmatrix}$. Since the 2-norm is invariant under orthogonal transformations we get: $\|Ax-b\|_2^2 = \| \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} (Ax-b)\|_2^2 = \| \begin{bmatrix} R \\ O \end{bmatrix} x - \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} b\|_2^2 = \|Rx - Q_1^T b\|_2^2 + \|Q_2^T b\|_2^2$.
This is as small as possible for $Rx = Q_1^T b$, which is the solution formula. The norm of the residual is then $\|Ax-b\|_2 = \|Q_2^T b\|_2$.

4a. $A^+ = V_1\Sigma_r^{-1}U_1^T$ where $A = U_1\Sigma_r V_1^T$ is the compact SVD of A.
4b. $AA^+ = U_1\Sigma_r V_1^T V_1\Sigma_r^{-1}U_1^T = U_1U_1^T$, which is clearly symmetric.
$A^+A = V_1\Sigma_r^{-1}U_1^T U_1\Sigma_r V_1^T = V_1V_1^T$ is symmetric too.
$AA^+A = U_1U_1^T U_1\Sigma_r V_1^T = U_1\Sigma_r V_1^T = A$
$A^+AA^+ = V_1V_1^T V_1\Sigma_r^{-1}U_1^T = V_1\Sigma_r^{-1}U_1^T = A^+$.
4c. If A has full rank then $A^+A = V_1V_1^T = I$, so $A^+ = A^{-1}$ and then
$A = Q_1R \Rightarrow A^+ = A^{-1} = R^{-1}Q_1^T$, because $A^+A = R^{-1}Q_1^TQ_1R = R^{-1}R = I$, since Q_1 has ortonormal columns and R is nonsingular.

5a) Use a Givens rotation $R(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$ to zero-out the (3,1) element:

$R(2,3,\theta)A = \begin{bmatrix} 3 & 0 & 2 \\ 2s & 4c+s & c+2s \\ -2c & -4s+c & -s+2c \end{bmatrix}$. By $s=1,\ c=0$ we get the desired $R(2,3,\theta)A =$ $\begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ 0 & -4 & -1 \end{bmatrix}$ and then $R(2,3,\theta)AR(2,3,\theta)^T = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 4 \end{bmatrix}$.

5b) For $H = I - 2uu^T$ first calculate $\hat{u} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$ and normalize to $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The Householder reflection becomes $H = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $HAH = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$.

5c) We apply spectral slicing on the tridiagonal matrix from b): $B = HAH = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, i.e. we want to find a factorization $B - \sigma I = LDL^T$ with $\sigma = 5$, so we should identify the elements in L and D from:

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ 0 & l_2 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_1 & 0 \\ 0 & 1 & l_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d_1 & l_1 d_1 & 0 \\ l_1 d_1 & d_2 + l_1^2 d_1 & l_2 d_2 \\ 0 & l_2 d_2 & d_3 + l_2^2 d_2 \end{bmatrix}.$$

We find $d_1 = -2,\ \ l_1 = -1,\ \ d_2 = -1,\ \ l_2 = -1$ and $d_3 = 0$. We use the fact that $B - 5I$ and D are congruent and have the same inertia. The eigenvalues of D are ≤ 0 so the eigenvalues of A are ≤ 5. One eigenvalue of D is 0 so one eigenvalue of A is 5.

6a) Let $x \in R(X)$ i.e. $x = Xz$ for some z. Then $Ax = AXz = XBz \in R(X)$ so X is right invariant subspace.

6b) Let λ be an eigenvalue of B. Then $By = \lambda y$ for some eigenvektor y and then $XBy = \lambda Xy \Rightarrow AXy = \lambda Xy$ so λ is also an eigenvalue of A with eigenvector Xy.

6c) Assume X_1 contains some eigenvectors of A as columns. Then X_1 is a right invariant subspace $AX_1 = X_1 D$, where D is diagonal with corresponding eigenvalues. Let now $X = [X_1\ X_2]$ be non-singular. Then $X^{-1}AX = X^{-1}[AX_1\ AX_2] = [X^{-1}X_1 D\ X^{-1}AX_2] =$ $\begin{bmatrix} D & \tilde{A_{12}} \\ O & \tilde{A_{22}} \end{bmatrix}$, so the rest of the eigenvalues of A are the eigenvalues of the smaller matrix $\tilde{A_{22}}$.

7. See text book or lecture notes.