Department of
Mathematics
Göteborg

**EXAMINATION FOR**
**NUMERICAL LINEAR ALGEBRA, TMA265/MMA600**
**2012-01-16**

**DATE: Monday January 16 TIME: 13.15 - 17.15 PLACE: Math Department**

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| Examiner: | Ivar Gustafsson, tel: 772 10 94 |
| Teacher on duty: | Ivar Gustafsson |
| Solutions: | Will be announced at the end of the exam on the board nearby room MVF21 |
| Result: | Will be sent to you by February 6 at the latest<br>Your marked examination can be received at the student's office<br>at Mathematics Department, daily 12.30-13 |
| Grades: | To pass requires 13 point, including bonus points from homework assignments<br>Grades are evaluated by a formula involving also the computer exercises |
| Aids: | None (except dictionaries) |

Instructions:
- State your methodology carefully. Motivate your statements clearly.
- Only write on one page of the sheet. Do not use a red pen. Do not answer more than one question per page.
- Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover.

**GOOD LUCK!**

**Question 1.**
Consider the least squares problem $min_x \|Ax - b\|_2$ with full rank and perturbation $\delta b$ to the right hand side $b \in R^m$ and no perturbation to the matrix $A \in R^{m \times n}$ with $m > n$. Show that the corresponding error $\delta x$ in the solution $x$ is bounded by $\frac{\|\delta x\|_2}{\|x\|_2} \leq \kappa(A) \frac{1}{\cos\theta} \frac{\|\delta b\|_2}{\|b\|_2}$ where $\theta$ is the angle between $b$ and $Ax$. **(3p)**
**Hint:** Apply the perturbation theory for linear systems to the normal equations. $\kappa(A)$ is the condition number $\|A\|_2 \|A^+\|_2$, where $A^+$ is a generalized inverse.

**Question 2.**
**a)** Describe matrix by matrix multiplication; C=A*B+C, based on blocking. Derive the ratio q of flops to memory references, when the matrices are n by n and the fast memory contains M words. **(2p)**
**b)** State a stable factorization of a symmetric indefinite real matrix. **(1p)**
**c)** Which factorization do you get in b) if $A$ is symmetric positive definite? **(1p)**

**Question 3**.
Define the concept: Normal equations of the second kind. State a problem which is solved by the normal equations of the second kind. Derive the solution! When is the solution unique? **(3p)**

**Question 4**.
Transform the matrix $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 4 & 2 & 0 & 1 \\ 3 & 4 & 0 & 1 \\ 0 & 3 & -4 & 1 \end{bmatrix}$ to upper Hessenberg form by a similar transformation based on a Givens rotation. **(3p)**

**Question 5**.
**a)** Define the concept: compact singular value decomposition (SVD) of a matrix. **(1p)**
**b)** Use SVD to express the projection of a vector on the rowspace of a matrix. **(2p)**

**Question 6**. Prove the following theorem by Bauer-Fike:
Let $A \in R^{n\times n}$ be diagonalizable: $X^{-1}AX = D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ and let $\mu$ be an eigenvalue of the matrix $A + E$, where $E \in R^{n\times n}$.
Then for any p-norm:

$$\min_{1\leq i\leq n} |\mu - \lambda_i| \leq \kappa_p(X)\|E\|_p$$

where $\kappa_p$ is the condition number in p-norm. **(3p)**

**Question 7**.
Perform the first step in the bidiagonalization of a matrix $A = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 1 & 2 & -1 & 2 \\ -1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ i.e. find Householder reflections H and Z such that $A_1 = HAZ$ has the form
$A_1 = \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}$. **(3p)**

**Question 8**.
**a)** Descibe briefly all steps in the method: Householder **explicit** shift QR iteration, for computing eigenvalues of a real unsymmetric matrix. **(2p)**
**b)** Prove that the Hessenberg form is preserved during the iterations in a). **(1p)**