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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination January 16, 2012

1) $\left.\begin{array}{r} A^TA(x+\delta x)=A^T(b+\delta b) \\ A^TAx=A^Tb \end{array}\right\} \Rightarrow A^TA\delta x = A^T\delta b \Rightarrow \delta x = (A^TA)^{-1}A^T\delta b = A^+\delta b$

$\Rightarrow \|\delta x\|_2 \le \|A^+\|_2\|\delta b\|_2 \Rightarrow \frac{\|\delta x\|_2}{\|x\|_2} \le \|A^+\|_2\frac{\|\delta b\|_2}{\|x\|_2} = \|A^+\|_2\|A\|_2\frac{\|b\|_2\|\delta b\|_2}{\|A\|_2\|b\|_2\|x\|_2} = \kappa(A)\frac{\|b\|_2\|\delta b\|_2}{\|A\|_2\|b\|_2\|x\|_2}$.
By the inequality $\|Ax\|_2 \le \|A\|_2\|x\|_2$ and the fact that $\cos\theta = \frac{\|Ax\|_2}{\|b\|_2}$ we now get
$\frac{\|\delta x\|_2}{\|x\|_2} \le \kappa(A)\frac{\|b\|_2\|\delta b\|_2}{\|Ax\|_2\|b\|_2} = \kappa(A)\frac{1}{\cos\theta}\frac{\|\delta b\|_2}{\|b\|_2}$.

2 a) See text book or lecture notes.
2 b) $A = LDL^T$, L lower triangular, D block diagonal with 1x1 or 2x2 diagonal blocks.
2 c) $A = LDL^T$ with D positive diagonal.

3) Normal equations of the second kind: $AA^Ty = b$, where $A \in R^{m\times n}$, $m < n$ and $b \in R^m$.
The problem: $\left\{\begin{array}{l} min_x\|x\|_2 \\ subject\ to\ Ax = b \end{array}\right.$,
If x_0 is a solution to $Ax = b$, then all $x = x_0 + x_h$, where $x_h \in N(A)$, are solutions, since $Ax = Ax_0 + Ax_h = Ax_0 = b$.
$\|x\|_2$ is minimized if $x \in N(A)^\perp = R(A^T)$ i.e. for $x = A^Ty$ for some $y \in R^m$ and then $b = Ax = AA^Ty$ i.e. the problem is solved by
1) $AA^Ty = b$, 2) $x = A^Ty$. The solution is unique if $rank(A) = m$.

4) Use a Givens rotation $R(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to zero-out the (3,1) element:

$R(2,3,\theta)A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 4c-3s & 2c-4s & 0 & c-s \\ 4s+3c & 2s+4c & 0 & s+c \\ 0 & 3 & -4 & 1 \end{bmatrix}$. By $s=-3/5$, $c = 4/5$ we get the desired

$R(2,3,\theta)A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 5 & 4 & 0 & 7/5 \\ 0 & 2 & 0 & 1/5 \\ 0 & 3 & -4 & 1 \end{bmatrix}$ and then

$$R(2,3,\theta)AR(2,3,\theta)^T = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 5 & 4 & 0 & 7/5 \\ 0 & 2 & 0 & 1/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4/5 & -3/5 & 0 \\ 0 & 3/5 & 4/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1/5 & 7/5 & 0 \\ 5 & 16/5 & -12/5 & 7/5 \\ 0 & 8/5 & -6/5 & 1/5 \\ 0 & 0 & -5 & 1 \end{bmatrix}.$$

5a) $A = U\Sigma V^T$, U and V with orthonormal columns and Σ diagonal $r \times r$, where $r = rank(A)$.
5b) $A^T = V\Sigma U^T$ with Σ and U "regular" matrices. Thus, $Col(A^T) = Col(V)$, so the columns of V make up an orthonormal basis for $Col(A^T) = Row(A)$ and the projection is then by the matrix VV^T.

6) Assume μ is not an eigenvalue of A, otherwise the result is trivial.
The matrix $A + E - \mu I$ is then singular and so is the matrix
$X^{-1}(A + E - \mu I)X = D - \mu I + X^{-1}EX$.
Then there exists a $z \neq 0$ such that $(D - \mu I)z = -X^{-1}EXz$.
Solving for z in this equation gives
$z = -(D - \mu I)^{-1}X^{-1}EXz$ and taking norms gives
$\|z\|_p = \|(D - \mu I)^{-1}X^{-1}EXz\|_p \leq$
$\|(D - \mu I)^{-1}\|_p\|X^{-1}\|_p\|E\|_p\|X\|_p\|z\|_p = \|(D - \mu I)^{-1}\|_p\kappa_p(X)\|E\|_p\|z\|_p$. (*)
But $D - \mu I$ is diagonal so $\|(D - \mu I)^{-1}\|_p = \frac{1}{\min_{1 \leq i \leq n} |\lambda_i - \mu|}$ for any norm p. Recall that $D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$
Divide the inequality (*) by $\|z\|_p$ to obtain the result.

7) For the construction of H we have $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow u = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$,

$H = I - 2uu^T$. Here u is orthogonal to columns 2-4 of A. Thus, $HA = \begin{bmatrix} 2 & 4 & 0 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

For Z we get $\tilde{u} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \Rightarrow u = \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \end{bmatrix}$, $Z = I - 2uu^T$

and $HA \times Z$ can by computed row by row, $HAZ = \begin{bmatrix} 2 & 5 & 0 & 0 \\ 0 & 2.8 & -1 & -0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 2.2 & 2 & 0.4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$.

8) See text book or lecture notes.