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Numerical Linear Algebra, TMA265/MMA600
Solutions to the examination 21 October 2011

1. Let $\begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} = QR$ be a QR-factorization. Then
$A^T A + \alpha I = \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix}^T \begin{bmatrix} A \\ \sqrt{\alpha} I \end{bmatrix} = (QR)^T(QR) = R^T Q^T Q R = R^T R.$

2. See text book or lecture notes.

3. Let $B^T = [Q \quad \tilde{Q}] \begin{bmatrix} R \\ 0 \end{bmatrix}$ be the full QR-factorization of B^T. Then $Bx = d \Leftrightarrow$ $[R^T \quad 0] \begin{bmatrix} Q^T \\ \tilde{Q}^T \end{bmatrix} x = d$. By the orthogonal variabel transformation $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Q^T x \\ \tilde{Q}^T x \end{bmatrix}$, we then get $x = Qy_1 + \tilde{Q} y_2$, where y_2 is free and $y_1 = R^{-T} d$ (formally).
Now, the over-determined system $Ax = b \Leftrightarrow AQR^{-T}d + A\tilde{Q}y_2 - b = 0$. In least squares sense, this problem is solved for y_2 by the compact QR-factorization of $A\tilde{Q} = Q_1 R_1$. The solution is then formally: $y_2 = R_1^{-1} Q_1^T (b - AQR^{-T}d)$.

4a. $A^+ = V_1 \Sigma_r^{-1} U_1^T$ where $A = U_1 \Sigma_r V_1^T$ is the compact SVD of A.
4b. $[A \ B]$ nonsingular $\Rightarrow$ A and B have full rank $\Rightarrow$ $A^+ = (A^T A)^{-1} A^T$, $\quad B^+ = (B^T B)^{-1} B^T$. Further $A^T B = O \Rightarrow A^+ B = (A^T A)^{-1} A^T B = 0$ and $A^T B = 0 \Rightarrow B^T A = O \Rightarrow B^+ A = (B^T B)^{-1} B^T A = O$. Finally, $\begin{bmatrix} A^+ \\ B^+ \end{bmatrix} [A \ B] = \begin{bmatrix} A^+ A & A^+ B \\ B^+ A & B^+ B \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I$.

5a. A and B are similar if there exists a nonsingular S such that $B = S^{-1}AS$.
$Ax = \lambda x \Leftrightarrow S^{-1}Ax = \lambda S^{-1}x \Leftrightarrow S^{-1}A(SS^{-1})x = \lambda S^{-1}x \Leftrightarrow S^{-1}AS(S^{-1}x) = \lambda(S^{-1}x) \Leftrightarrow B(S^{-1}x) = \lambda(S^{-1}x)$.

5b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 4c-3s & c & -2s \\ 4s+3c & s & 2c \end{bmatrix}$.

Choose $c = 4/5,\ s = -3/5$ to zero-out the (3,1)-element.

$$\begin{bmatrix} 2 & 4 & 3 \\ 5 & 4/5 & 6/5 \\ 0 & -3/5 & 8/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 5 & 34/25 & 12/25 \\ 0 & 12/25 & 41/25 \end{bmatrix}.$$

6a. Small errors in data (the input) result in small errors in the answer to the problem (the output).

6b. The algorithm gives the exact solution to a slightly pertubed problem.

6c. $(A+E)\bar{x} = A\bar{x} + E\bar{x} = \bar{\lambda}\bar{x} + r + E\bar{x}$, which equals $\bar{\lambda}\bar{x}$ if $E\bar{x} = -r$ and then $(\bar{\lambda}, \bar{x})$ is an eigenpair of $A+E$. This equation holds for $E = -\frac{r\bar{x}^T}{\bar{x}^T\bar{x}}$ and then $\|E\|_2 = \frac{\|r\|_2}{\|\bar{x}\|_2}$ since $\|r\bar{x}^T\|_2 = max_{y\neq 0}\frac{\|r\bar{x}^Ty\|_2}{\|y\|_2} = max_{y\neq 0}\frac{|\bar{x}^Ty|\|r\|_2}{\|y\|_2} = \|\bar{x}\|_2\|r\|_2$, where the maximum is attained for $y = \bar{x}$.

7a. $V^TAV = T$ where T is real, block-upper triangular with diagonal blocks of size 1×1 or 2×2.

7b. From the eigenvalues of the 2×2 diagonal blocks.

7c. It converges to an upper triangular matrix with the eigenvalues (real and complex) on the diagonal.

8. See text book or lecture notes.