Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: October 23, Time: 8.30 - 12.30, Place: at CTH, Maskinhuset

Question 1

- 1. Find eigenvalues and eigenvectors for a matrix A such that

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

  (1p)
- 2. Determine if the matrix A is positive-definite or not. Explain why. **(1p)**
- 3. Find conjugate transpose matrix A^* to the matrix A. **(1p)**
- 4. Find inverse matrix A^{-1} to the matrix A via the matrix of cofactors. **(1p)**

Question 2

- 1. Describe all steps in the algorithm of Gaussian elimination to solve a system of linear equation $Ax = b$. **(2p)**
- 2. Why application of the LU decomposition without pivoting using Gaussian elimination of the matrix A

$$\mathbf{A} = \begin{bmatrix} 10^{-7} & 1 \\ 1 & 1 \end{bmatrix}$$

  will give rise of numerical instability in the computed solution $\tilde{x}$ of a linear system of equations $Ax = b$? What happens when we will apply LU factorization with pivoting for matrix A? Compare the condition number of the matrix A given by $k(A) = ||A||_\infty \cdot ||A^{-1}||_\infty$ and the condition numbers of the matrices L and U given by $k(L) = ||L||_\infty \cdot ||L^{-1}||_\infty$ and $k(U) = ||U||_\infty \cdot ||U^{-1}||_\infty$ in both cases. **(2p)**

Question 3

- 1. Consider the solution of a linear system $Ax = b$. Derive bounds for the error $||\delta x||$ in the computed solution $\tilde{x} = \delta x + x$ using facts that the matrix A is given with an error δA and the right hand side b is given with an error δb. Use obtained inequality to get expression for condition number $k(A)$ of the matrix A. **(2p)**
- 2. Consider the solution of a linear system $Ax = b$. Derive bounds for the error $||\delta x||$ in the computed solution $\tilde{x} = \delta x + x$ using definition of the residual $r = A\tilde{x} - b$. **(2p)**

Question 4

- 1. Let A will be $m \times n$ matrix with $m \geq n$. Describe main concept of the QR decomposition of a matrix A. **(2p)**

- 2. Let A will be $m \times n$ matrix with $m \geq n$. Derive the formula for the solution x of a linear system $Ax = b$ that minimizes $||Ax - b||_2$ using the QR decomposition of the matrix $A = QR$ and the fact that the vectors $Q(Rx - Q^T b)$ and $(I - QQ^T)b$ are orthogonal. **(2p)**

Question 5

- 1. Let A will be $N \times N$ matrix with elements a_{ij} and A^* be the conjugate transpose matrix to a matrix A. Using definition of the scalar product $(\cdot,\cdot)$ for the vector f as $(f,f) = \sum_{i=1}^{N} f_i \bar{f}_i$ prove that

$$(Ax, y) = (x, A^* y)$$

  (1p)
- 2. Let A will be $N \times N$ matrix and A^* be the conjugate transpose matrix to a matrix A. Let B will be $N \times N$ matrix and B^* be the conjugate transpose matrix to a matrix B.

  Prove

$$[A \cdot B]^* = B^* A^*.$$

  (1p)
- 3. Transform the matrix A to the tridiagonal form using Householder reflection.

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

  (1p)
- 4. Transform the matrix A to the tridiagonal form using Given's rotation.

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{bmatrix}$$

(1p)

Question 6

- 1. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Prove that the eigenvalues of the symmetric matrix $A^T A$ are σ_i^2, and the right singular vectors v_i (columns of V) are corresponding orthonormal eigenvectors. **(1p)**
- Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Prove that if A has full rank, the solution of $\min_x \|Ax - b\|_2$ is $x = V\Sigma^{-1}U^T b$. **(2p)**

Question 7

- 1. Let entries of the matrix A are schematically given by

$$\mathbf{A} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x. \end{bmatrix}$$

  Here, x denotes nonzero entry of the matrix A. Describe how to compute QR decomposition of a 4×4 matrix A using Householder transformation (reflection). **(2p)**
- 2. Let entries of the real matrix A are defined as in item 1. Using Hessenberg reduction algorithm describe how to obtain for a given real matrix A an upper Hessenberg matrix A_2 such that

$$\mathbf{A_2} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x. \end{bmatrix}$$

  (2p)

Numerical Linear Algebra

TMA265/MMA600
Solutions to the examination at 23 October 2012

Question 1

1. We should solve characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = 0.$$

Solving above equation for λ we get two eigenvalues $\lambda_1 = 5, \lambda_2 = 1$.

To find eigenvectors x we need to solve equation $Ax = \lambda x$ for x and for λ_1 and λ_2. We have: for $\lambda_1 = 5$ any vector satisfying to $x_1 = x_2$ will be eigenvector; and for $\lambda_1 = 1$ any vector satisfying to $x_1 = -x_2$ will be eigenvector.

2. The matrix A is positive-definite if $x^T Ax > 0$. We have

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + (x_1 + x_2)^2 + (x_1 + x_2)^2 + x_2^2 > 0$$

Reconfirm that: Since A is real symmetric matix then we can consider their determinants: $D_1 = 3 > 0$, $D_2 = 9 - 4 = 5 > 0$. Since all determinants are positive then A is positive definite.

3. We use definition of A^*:

$$A^* = \overline{A^T}.$$

$$A^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \overline{A^T} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Thus, $A = A^*$.

4. By definition of an inverse matrix we have:

$$A^{-1} = \frac{1}{\det A} (C^T)_{ij}$$

Thus,

$$C = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix},$$

$$A^{-1} = \frac{1}{\det A} (C^T)_{ij} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}.$$

Question 2

See Lecture 3 and the course book.

Question 3

See Lecture 3 and the course book.

Question 4

1. See Lecture 6 and the course book.
2. Write $Ax - b$ as

$$\begin{aligned} Ax - b &= QRx - b = QRx - (QQ^T + I - QQ^T)b \\ &= Q(Rx - Q^T b) - (I - QQ^T)b. \end{aligned}$$

Note that the vectors $Q(Rx - Q^T b)$ and $(I - QQ^T)b$ are orthogonal, because $(Q(Rx - Q^T b))^T((I - QQ^T)b) = (Rx - Q^T b)^T[Q^T(I - QQ^T)]b = (Rx - Q^T b)^T[0]b = 0$. Therefore, by the Pythagorean theorem,

$$\begin{aligned} \|Ax - b\|_2^2 &= \|Q(Rx - Q^T b)\|_2^2 + \|(I - QQ^T)b\|_2^2 \\ &= \|Rx - Q^T b\|_2^2 + \|(I - QQ^T)b\|_2^2. \end{aligned}$$

This sum of squares is minimized when the first term is zero, i.e., $x = R^{-1}Q^T b$.

Question 5

1. Since $A^* = \overline{A^T}$ we can write with defining matrix $B = A^*$ such that $b_{ij} = \overline{a_{ji}}$

$$\begin{aligned} (Ax, y) &= \sum_{i=1}^{N}(\sum_{j=1}^{N} a_{ij}x_j)\overline{y_i} = \sum_{j=1}^{N} x_j(\sum_{i=1}^{N} \overline{\overline{a_{ij}}y_i}) \\ &= \sum_{j=1}^{N} x_j(\sum_{i=1}^{N} \overline{b_{ji}y_i}) = \sum_{i=1}^{N} x_i(\overline{\sum_{j=1}^{N} b_{ij}y_j}) = (x, By) = (x, A^*y). \end{aligned}$$

2. By definition of the conjugate transpose matrix we have

$$[A \cdot B]^* = \overline{(A \cdot B)^T} = \overline{B^T \cdot A^T} = \overline{B^T} \cdot \overline{A^T} = B^* A^*$$

3. To obtain tridiagonal matrix from the matrix A $= \begin{bmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{bmatrix}$ using Householder transformation we make following steps:

- Step1 . First compute α as

$$\alpha = -\mathrm{sgn}(a_{21})\sqrt{\sum_{j=2}^{n} a_{j1}^2} = -\sqrt{(a_{21}^2 + a_{31}^2)} = -\sqrt{4^2 + 3^2} = -5.$$

- Step 2. Using α we find r as

$$r = \sqrt{\frac{1}{2}(\alpha^2 - a_{21}\alpha)} = \sqrt{\frac{1}{2}((-5)^2 - 4 \cdot (-5))} = \frac{3\sqrt{5}}{\sqrt{2}}.$$

- Step 3. Then we compute components of vector v:

$$\begin{aligned} v_1 &= 0, \\ v_2 &= \frac{a_{21} - \alpha}{2r} = \frac{3\sqrt{2}}{2\sqrt{5}}, \\ v_3 &= \frac{a_{31}}{2r} = \frac{\sqrt{2}}{2\sqrt{5}}. \end{aligned}$$

and we have

$$v^{(1)} = \begin{bmatrix} 0 \\ \frac{3\sqrt{2}}{2\sqrt{5}} \\ \frac{\sqrt{2}}{2\sqrt{5}} \end{bmatrix},$$

- Step 4 . Then compute matrix P^1

$$P^1 = I - 2v^{(1)}(v^{(1)})^T$$

to get $\mathrm{P}^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4/5 & -3/5 \\ 0 & -3/5 & 4/5 \end{bmatrix}$

- Step 5.
After that we can obtain tridiagonal matrix $A^{(1)}$ as

$$A^{(1)} = P^1AP^1 = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 7.32 & -0.76 \\ 0 & -0.76 & 5.68. \end{bmatrix}$$

4. To obtain tridiagonal matrix from the matrix
$\mathrm{A} = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{bmatrix}$
using Given's rotation we have to zero out $(3,1)$ and $(1,3)$ elements of the matrix A. Thus we use the Given's rotation $R(2,3,\theta)$ such that

$$R(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

We compute

$$R(2,3,\theta) \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 \\ 4c-3s & 6c-s & c-7s \\ 4s+3c & 6s+c & s+7c \end{bmatrix}$$

Element $(3,1)$ of the matrix will be zero if $4s + 3c = 0$. This is true when $c = 4/5$ and $s = -3/5$. To compute c, s we have used formulas:

$$\begin{aligned} r &= \sqrt{a^2+b^2} = \sqrt{4^2+3^2} = 5, \\ c &= \frac{a}{r} = 4/5, \\ s &= \frac{-b}{r} = -3/5. \end{aligned}$$

Next, to get tridiagonal matrix we have to do :

$$R(2,3,\theta) \cdot AR(2,3,\theta)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 3/5 \\ 0 & -3/5 & 4/5 \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 & 3 \\ 4 & 6 & 1 \\ 3 & 1 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 0 \\ 5 & 7.32 & 0.76 \\ 0 & 0.76 & 5.68 \end{bmatrix}$$

Question 6

- 1. Write $A^TA = V\Sigma U^TU\Sigma V^T = V\Sigma^2V^T$. This is an eigendecomposition of A^TA, with the columns of V the eigenvectors and the diagonal entries of Σ^2 the eigenvalues.
- 2. Let us consider
$\|Ax - b\|_2^2 = ||U\Sigma V^Tx - b||_2^2$. Since A has full rank, so does Σ, and thus Σ is invertible. Now let $[U, \tilde{U}]$ be square and orthogonal as above so

$$\begin{aligned} ||U\Sigma V^Tx - b||_2^2 &= \left\| \begin{bmatrix} U^T \\ \tilde{U}^T \end{bmatrix} (U\Sigma V^Tx - b) \right\|_2^2 \\ &= \left\| \begin{bmatrix} \Sigma V^Tx - U^Tb \\ -\tilde{U}^Tb \end{bmatrix} \right\|_2^2 \\ &= ||\Sigma V^Tx - U^Tb||_2^2 + \|\tilde{U}^Tb\|_2^2. \end{aligned}$$

This is minimized by making the first term zero, i.e., $x = V\Sigma^{-1}U^Tb$.

Question 7

- 1. We show how to compute the QR decomposition of a 4-by-4 matrix A using Householder transformations. This example will make the pattern for general m-by-n matrices evident. In the matrices below, P_i is an orthogonal matrix, x denotes a generic nonzero entry, and o denotes a zero entry.
  1. Choose P_1 so
$$A_1 \equiv P_1A = \begin{bmatrix} x & x & x & x \\ o & x & x & x \\ o & x & x & x \\ o & x & x & x \end{bmatrix}.$$
  2. Choose $P_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_2' \end{array}\right]$ so
$$A_2 \equiv P_2A_1 = \begin{bmatrix} x & x & x & x \\ o & x & x & x \\ o & o & x & x \\ o & o & x & x \end{bmatrix}.$$
  3. Choose $P_3 = \left[\begin{array}{cc|c} 1 & & \\ & & 0 \\ & 1 & \\ \hline & 0 & P_3' \end{array}\right]$ so
$$A_3 \equiv P_3A_2 = \begin{bmatrix} x & x & x & x \\ o & x & x & x \\ o & o & x & x \\ o & o & o & x \end{bmatrix}.$$

  Here, we have chosen a Householder matrix P_i' to zero out the subdiagonal entries in column i; this does not disturb the zeros already introduced in previous columns.

  Let us call the final 4-by-4 upper triangular matrix $\tilde{R} \equiv A_3$. Then $A = P_1^TP_2^TP_3\tilde{R} = QR$, where Q is the first three columns of $P_1^TP_2^TP_3^T = P_1P_2P_3$ (since all P_i are symmetric) and R is the first four rows of $\tilde{R}$.
- 2. Given a real matrix A, we seek an orthogonal Q so that QAQ^T is upper Hessenberg.

1. Choose Q_1 so

$$Q_1 A = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \text{ and } A_1 \equiv Q_1 A Q_1^T = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}.$$

Q_1 leaves the first row of $Q_1 A$ unchanged, and Q_1^T leaves the first column of $Q_1 A Q_1^T$ unchanged, including the zeros.

2. Choose Q_2 so

$$Q_2 A_1 = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \text{ and } A_2 \equiv Q_2 A_1 Q_2^T = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}.$$

Q_2 changes only the last two rows of A_1, and Q_2^T leaves the first two columns of $Q_2 A_1 Q_2^T$ unchanged, including the zeros.

Altogether $A_2 = (Q_2 Q_1) \cdot A (Q_2 Q_1)^T \equiv Q A Q^T$ is upper Hessenberg matrix.