Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: January 17 2013, Time: 8.30 - 12.30, Place: at CTH, Maskinhuset

Question 1

- 1. Find eigenvalues for a rotation matrix A such that

$$\mathbf{A} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}$$

  (1p)
- 2. Find inverse matrix A^{-1} to the matrix A via the matrix of cofactors. **(1p)**
- 3. Determine if the matrix A is orthogonal or not, explain why. **(1p)**

Question 2

- 1. Describe the main idea of Cholesky factorization of the matrix A for solution of a system of linear equation $Ax = b$. **(2p)**
- 2. Prove that if there exists a unique lower triangular nonsingular matrix L with positive diagonal entries such that $A = LL^T$ then A is symmetric positive definite matrix.

  (2p)

Question 3

- 1. Let $Ax = b$, the matrix A is given with an error δA, the right hand side b is given with an error δb and the computed solution $\tilde{x}$ is such that $\tilde{x} = \delta x + x$. Derive inequality for the relative change of $\frac{\|\delta x\|}{\|\tilde{x}\|}$ in terms of the condition number of the matrix A and relative change $\frac{\|\delta A\|}{\|A\|}$ in the data. **(2p)**
- 2. Let A be the diagonal matrix and B any nonsingular matrix. Prove that $k(AB) = k(B)$, where $k(AB)$ is the condition number of AB and $k(B)$ is the condition number of B. **(2p)**

Question 4

- 1. Derive normal equations $A^T Ax = A^T b$ in the method of normal equations. **(2p)**
- 2. Why $x = (A^T A)^{-1} A^T b$ is minimizer of $\|Ax - b\|_2^2$ in the method of normal equations? **(2p)**
- 3. Get formula $x = R^{-1} Q^T b$ for x that solves $Ax = b$ using the solution of the normal equations $x = (A^T A)^{-1} A^T b$ and QR decomposition of the matrix $A = QR$. **(2p)**

Question 5

- 1. Transform the matrix A to the tridiagonal form using Householder reflection.

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

  (2p)

- 2. Transform the matrix A to the tridiagonal form using Given's rotation.

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

(2p)

Question 6

- 1. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Prove that $\|A\|_2 = \sigma_1$ and $\|A^{-1}\|_2^{-1} = \sigma_n$. Here $\sigma_1 \geq ... \geq \sigma_n \geq 0$ are singular values of Σ. **(1p)**
- 2. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Prove that the eigenvalues of the symmetric matrix AA^T are σ_i^2 and $m - n$ zeros.
  (1p)
- 3. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Let m-by-n pixel image corresponds to the matrix A. Write formula for compressed image A_k which will be the best rank-k approximation of the matrix A.
  (1p)

Question 7

- 1. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A with $m \geq n$. Define the Moore-Penrose pseudoinverse matrix A^+ of the matrix A.
  (1p)
- 2. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A with $m \geq n$. Using definitions of A and A^+ prove that $AA^+A = A$.
  (2p)

Numerical Linear Algebra

TMA265/MMA600
Solutions to the examination at 17 January 2013

Question 1

1. We should solve characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} \cos\Theta - \lambda & -\sin\Theta \\ \sin\Theta & \cos\Theta - \lambda \end{bmatrix} = 0.$$

Solving above equation for λ we get two eigenvalues $\lambda_1 = \cos\Theta + i\sin\Theta, \lambda_2 = \cos\Theta - i\sin\Theta$.

2. By definition of an inverse matrix we have:

$$A^{-1} = \frac{1}{\det A}(C^T)_{ij}$$

Thus,

$$C = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix},$$

$$C^T = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix},$$

$$A^{-1} = \frac{1}{\det A}(C^T)_{ij} = \begin{bmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{bmatrix}.$$

3. The rotation matrix A is the orhogonal matrix since $A \cdot A^{-1} = I$.

Question 2

1. See Lecture 5 and the course book.
2. Since $A = LL^T$ then $x^T Ax = (x^T L)(L^x) = \|L^T x\|_2^2 > 0$ for all $x \neq 0$ and thus A is s.p.d.

Question 3

1. See Lecture 3 and the course book Chapter 2.2.
2. We can write

$$k(AB) = \||(AB)^{-1}| \cdot |(AB)|\| = \||B^{-1}A^{-1}| \cdot |AB|\| = \||B^{-1}| \cdot |B|\| = k(B).$$

Question 4

1. See Lecture 5 and Chapter 3.2.1 of the course book.
1. See Lecture 5 and Chapter 3.2.1 of the course book.
2. Write

$$x = (A^T A)^{-1} A^T b = (R^T Q^T Q R)^{-1} R^T Q^T b = (R^T R)^{-1} R^T Q^T b = R^{-1} R^{-T} R^T Q^T b = R^{-1} Q^T b.$$

Question 5

1. To obtain tridiagonal matrix from the matrix A $= \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ using Householder transformation we make following steps:

- Step1 . First compute α as

$$\alpha = -\text{sgn}(a_{21})\sqrt{\sum_{j=2}^{n} a_{j1}^2} = -\sqrt{(a_{21}^2 + a_{31}^2)} = -\sqrt{0^2+2^2} = -2.$$

- Step 2. Using α we find r as

$$r = \sqrt{\frac{1}{2}(\alpha^2 - a_{21}\alpha)} = \sqrt{\frac{1}{2}(4+0)} = \sqrt{2}.$$

- Step 3. Then we compute components of vector v:

$$\begin{aligned} v_1 &= 0, \\ v_2 &= \frac{a_{21}-\alpha}{2r} = \frac{1}{\sqrt{2}}, \\ v_3 &= \frac{a_{31}}{2r} = \frac{1}{\sqrt{2}} \end{aligned}$$

  and we have

$$v^{(1)} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

- Step 4 . Then compute matrix P^1

$$P^1 = I - 2v^{(1)}(v^{(1)})^T$$

  to get $\mathrm{P}^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

- Step 5.
After that we can obtain tridiagonal matrix $A^{(1)}$ as

$$A^{(1)} = P^1 A P^1 = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 2. \end{bmatrix}$$

2. To obtain tridiagonal matrix from the matrix
$\mathrm{A} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$
using Given's rotation we have to zero out $(3,1)$ and $(1,3)$ elements of the matrix A. Thus we use the Given's rotation $R(2,3,\theta)$ such that

$$R(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

We compute

$$R(2,3,\theta) \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ -2s & 2c-s & c-3s \\ 2c & 2s+c & s+3c \end{bmatrix}$$

Element $(3,1)$ of the matrix will be zero if $2c = 0$. This is true when $c = 0$. To compute c, s we can use

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get formulas:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} = \sqrt{0^2 + 2^2} = 2, \\ c &= \frac{a}{r} = 0, \\ s &= \frac{-b}{r} = -1. \end{aligned}$$

Next, to get tridiagonal matrix we have to do :

$$R(2,3,\theta) \cdot AR(2,3,\theta)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Question 6

- 1. See Lecture 7 and Theorem 3.3 of the course book.
- 2. See Lecture 7 and Theorem 3.3 of the course book.
- 3. Since $A = U\Sigma V^T$ then by Theorem 3.3 of the course book $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ will be the best rank-k approximation of the matrix A. A_k will represent the compressed image of A.

Question 7

- 1. Let A be the m-by-n matrix with $m \geq n$ and has a full rank such that $A = U\Sigma V^T$. Then Moore-Penrose pseudoinverse of A is $A^+ = (A^T A)^{-1} A^T$. If $m < n$ then $A^+ = A^T (AA^T)^{-1}$.
- 2.

$$\begin{aligned} AA^+A &= U\Sigma V^T (V\Sigma U^T U\Sigma V^T)^{-1} V\Sigma U^T U\Sigma V^T \\ &= U\Sigma V^T (V^{-1}\Sigma^{-2} V^{-T})\Sigma^2 = U\Sigma V^{-1} = U\Sigma V^T = A. \end{aligned}$$