Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: 2014 January 16, Time: 8.30 - 12.30, Place: at CTH, Maskinhuset

Question 1

- 1. Find eigenvalues for a matrix A which is defined as

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

  (1p)
- 3. Find inverse matrix A^{-1} to the matrix A defined as

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

  via the matrix of cofactors. **(2p)**
- 2. Compute $||A||_\infty, ||A||_1$ for matrix A defined as

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 5 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

  Find conjugate transpose matrix A^* to this matrix A. **(1p)**

Question 2

- 1. Describe the main idea of Cholesky factorization of the matrix A for solution of a system of linear equation $Ax = b$. **(2p)**
- 2. Prove that if there exists a unique lower triangular nonsingular matrix L with positive diagonal entries such that $A = LL^T$ then A is symmetric positive definite matrix.
  (2p)

Question 3

- 1. Derive the relative condition number $k_{CR}(A)$ of the matrix A (called also as Bauer condition number or Skeel condition number). **(2p)**
- 2. Let the matrix A is defined as

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix}$$

  and the vector $b = [a_{11}, 1]^T$ is such that $x = A^{-1}b$. Determine $k_{CR}(A)$.
  (2p)

Question 4

- 1. Give definitions for three main methods (normal equations, QR decomposition and SVD) which solve the linear least squares problem.
  (2p)
- 2. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Prove that the eigenvalues of the symmetric matrix AA^T are σ_i^2 and $m-n$ zeros.
  (2p)
- 3. Let $A = U\Sigma V^T$ be the SVD decomposition of the m-by-n matrix A, where $m \geq n$. Let m-by-n pixel image corresponds to the matrix A. Write formula for compressed image A_k which will be the best rank-k approximation of the matrix A.
  (2p)

Question 5

- Compute QR decomposition of the matrix A using Householder reflections:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

  (2p)
- Compute QR decomposition of the matrix A using Given's rotations

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

  (1p)

Question 6

- 1. Give the definition for a matrix pencil, regular and singular pencils.
  (1p)
- 2. Let $A - \lambda B$ is

$$\mathbf{A} - \lambda\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  Determine eigenvalues of $A - \lambda B$.
  (2p)

Question 7

- 1. Write algorithm for the Power method to find largest eigenvalue of the matrix A and the corresponding eigenvector. **(1p)**
- 2. Write Inverse iteration algorithm. Describe difference between this algorithm and the Power method.
  (2p)

Numerical Linear Algebra

TMA265/MMA600
Solutions to the examination at 16 January 2014

Question 1

1. We should solve characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = 0.$$

Solving above equation for λ we get three eigenvalues $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 5$ which are solutions to the equation $(2-\lambda)(-5\lambda + \lambda^2) = 0$.

2. By definition of an inverse matrix we have:

$$A^{-1} = \frac{1}{\det A}(C^T)_{ij}$$

Thus,

$$C = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -2 \\ -5 & 3 & 10 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 0 & 1 & -5 \\ 0 & 0 & 3 \\ 3 & -2 & 10 \end{bmatrix},$$

and thus $A^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 0 & 1 & -5 \\ 0 & 0 & 3 \\ 3 & -2 & 10 \end{bmatrix}$.

3. We use definition of A^*:

$$A^* = \overline{A^T}.$$

$$A^T = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

$||A||_1 = max(5, 6, 1) = 6, ||A||_\infty = max(3, 8, 1) = 8.$

Question 2

1. See Lecture 5 and the course book.

2. Since $A = LL^T$ then $x^T Ax = (x^T L)(L^x) = \|L^T x\|_2^2 > 0$ for all $x \neq 0$ and thus A is s.p.d.

Question 3

1. See Lectures 4 and the course book: $k_{CR}(A) = |||A^{-1}| \cdot |A|||$ is the componentwise relative condition number of A.

2. Since $|A^{-1}| \cdot |A| = I$ then $k_{CR}(A) = 1$.

Question 4

1. See Lecture 7 and the course book.
2. See Lecture 7 and Theorem 3.3 of the course book.
3. Since $A = U\Sigma V^T$ then by Theorem 3.3 of the course book $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ will be the best rank-k approximation of the matrix A. A_k will represent the compressed image of A.

Question 5

- 1. We only need to zero the (3, 2) entry.
  Take the $(1,1)$ minor

$$A' = M_{11} = \begin{pmatrix} 4 & 0 \\ 3 & 3 \end{pmatrix}.$$

and apply Hausholder transformation to get:

$$\mathbf{u} = \mathbf{x} + \alpha \mathbf{e}_1,$$

where $\mathbf{x} = (4,3)^T$, $\alpha = -sign(4) \cdot ||x||$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Here,

$$\alpha = -5.$$

Therefore

$$\mathbf{u} = (-1,3)^T, \quad ||u|| = \sqrt{10}.$$

and $\mathbf{v} = \frac{1}{\sqrt{10}}(-1,3)^T$, and then

$$P_2' = I - \frac{2}{\sqrt{10}^2} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}.$$

Then the matrix of the Householder transformation is

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & 0.6 & -0.8 \end{pmatrix}$$

Now, we find

$$R = P_2 A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 1.8 \\ 0 & 0 & -2.4 \end{pmatrix}.$$

The matrix P is orthogonal and R is upper triangular, so $A = QR$ is the required QR-decomposition with $P = P_2^T$.

- 2. To obtain QR decomposition of the matrix A

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

using Given's rotation we have to zero out $(3,2)$ element of the matrix A.

We construct Given's matrix G_2 in order to zero out $(3,2)$ element of the matrix A.

To do that we compute c, s from the known $a = 4$ and $b = 3$ as

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get formulas:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5, \\ c &= \frac{a}{r} = 0.8, \\ s &= \frac{-b}{r} = -0.6. \end{aligned}$$

The second Given's matrix will be

$$\mathbf{G_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

or

$$\mathbf{G_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{bmatrix}$$

Then upper triangular matrix R in the QR decomposition will be

$$\mathbf{R} = \mathbf{G_2} \cdot \mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 1.8 \\ 0 & 0 & 2.4 \end{bmatrix}$$

Then $A = G_2^T \cdot R = QR$ will be QR decomposition of the matrix A with $Q = G_2^T$.

Question 6

- 1. See Lecture 11 and the course book.
- 2. Since

$$A - \lambda B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} - \lambda \begin{bmatrix} 2 & & \\ & 0 & \\ & & 1 \end{bmatrix}.$$

Then $p(\lambda) = \det(A - \lambda B) = (1 - 2\lambda) \cdot (1 - 0\lambda) \cdot (0 - \lambda) = (2\lambda - 1)\lambda$, so the eigenvalues are $\lambda = \frac{1}{2}$, 0 and ∞. $\diamond$

Question 7

- 1. See Lecture 11 and the course book:
  Alg. Power method: Given x_0, we iterate

$$\begin{aligned} & i = 0 \\ & \textit{repeat} \\ & \quad y_{i+1} = Ax_i \\ & \quad x_{i+1} = y_{i+1}/||y_{i+1}||_2 \quad (\textit{approximate eigenvector}) \\ & \quad \tilde{\lambda}_{i+1} = x_{i+1}^T A x_{i+1} \qquad (\textit{approximate eigenvalue}) \\ & \quad i = i + 1 \\ & \textit{until convergence} \end{aligned}$$

- 2. . See Lecture 11 and the course book:
  We apply the power method to $(A - \sigma I)^{-1}$ instead of A, where σ is called a *shift*. This will let us converge to the eigenvalue closest to σ, rather than just λ_1. This method is called *inverse iteration* or the *inverse power method*.

Alg. Inverse iteration: Given x_0, we iterate

$$
\begin{array}{ll}
i = 0 & \\
\textit{repeat} & \\
\quad y_{i+1} = (A - \sigma I)^{-1} x_i & \\
\quad x_{i+1} = y_{i+1}/||y_{i+1}||_2 & (\textit{approximate eigenvector}) \\
\quad \tilde{\lambda}_{i+1} = x_{i+1}^T A x_{i+1} & (\textit{approximate eigenvalue}) \\
\quad i = i + 1 & \\
\textit{until convergence} &
\end{array}
$$

$A = S\Lambda S^{-1}$ implies $A - \sigma I = S(\Lambda - \sigma I)S^{-1}$ and so $(A - \sigma I)^{-1} = S(\Lambda - \sigma I)^{-1}S^{-1}$. Thus $(A - \sigma I)^{-1}$ has the same eigenvectors s_i as A with corresponding eigenvalues $((\Lambda - \sigma I)^{-1})_{jj} = (\lambda_j - \sigma)^{-1}$. We expect that x_i to converge to the eigenvector corresponding to the largest eigenvalue in absolute value.