Numerical Linear Algebra
Computer Labs

Notes

- To pass this course you should do one of the 4 computer assignments.
- You can work in groups by 2 persons
- Sent final report with your assignment to my e-mail. Report should have description of used techniques, tables and figures confirming your investigations. Attach also corresponding programs in Matlab or PETSc for testing and confirming your results.

Computer exercise 1 (3 p.)

Solve question Q1.20 in the text book

1. Question Q1.20, part 1.

Notes: program polyplot.m is also available to download from the homepage of our course.

Program polyplot.m forms the coefficients of a polynomial specified by its roots, and repeatedly add small perturbations to the coefficients, plotting the resulting perturbed roots.

Inputs parameters in the program polyplot.m:

r = vector of polynomial roots

e = maximum relative perturbation to make to each coefficient

m = number of random polynomials to generate

Output of this program is to plot of perturbed roots and generate polynomial coefficients. This will be first part of your assignment.

Notes to part 1: when you want to test your own inputs for roots (for example, test comples conjugate roots) use function **roots(C)** in matlab in order to find polynomial roots which can be used as input in the program polyplot.m

The function roots(C) in Matlab computes the roots of the polynomial whose coefficients are the elements of the vector C. If C has N+1 components, the polynomial is $p(X) = C(1) * X^N + ... + C(N) * X + C(N+1)$.

2. Question Q1.20, part 2.

For the second part of your assignment You need to compute the condition number c_i of every root. Use following formula to compute this condition number:

$$c_i = \frac{\sum_{i=0}^{d} |a_i x^i|}{|p'(x_i)|}, \tag{1}$$

where $p(x) = \sum_{i=0}^{d} a_i x^i$, where d is dimension of your polynomial. You need to modify program polyplot.m in order to plot 2 kinds of circles centered at $r(i)$ for every root. First, you need to plot circles with the radius which was computed using the formula (1). Second, you need to plot circles with biggest distance between perturbed and original root. Explain what you observe.

3. Question Q1.20, part 3. In this question you should show (present some numerical examples or some error output in matlab) that formula (1) "blows up" when $p'(r_i) = 0$, where $r(i)$ is i-th root of the polynomial. You also need to show (analytically and in the form of table or pictures) that roots of the perturbed polynomial differ from multiple root as $|\varepsilon|^{\frac{1}{m}}$, where m is multiplicity of the root and ε is accuracy.

To show that the perturbated roots differ from $r(i)$ by factor $|\varepsilon|^{\frac{1}{m}}$ you should use fact that the polynomial $p(x)$ can be written as

$$p(x) = q(x)(x - r(i))^m, \tag{2}$$

and the slightly perturbed polynomial can be written as $p(x) - q(x)\varepsilon$.

- 1. Write discretization of the Poisson's equation in two dimensions

$$- a\triangle u(x,y) = f(x,y) \tag{3}$$

on the unit square $\{(x,y) : 0 < x, y < 1\}$ with boundary conditions $u = 0$ on the boundary of this square. Here, the coefficient $a(x,y)$ is such that

$$a(x,y) = 1 + A(\sin(\frac{\pi}{3}x))^2 \cdot (\sin(\frac{\pi}{3}y))^2 \tag{4}$$

with values of the amplitude $A = 3, 12, 26$.

We produce the mesh with the points (x_i, y_j) such that $x_i = ih; y_j = jh$ with $h = 1/(N+1)$, where $N+1$ is the number of points in x and y directions. You should derive the equation

$$a_{i,j}(4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}) = h^2 f_{i,j} \tag{5}$$

- Write equation (5) as a single matrix equation in the form $Au = f$ with explicit entries of A.

- 2. Implement Algorithm 2.3 of the text book and solve the linear system of equations $Au = f$ by the Gaussian elimination for the matrix A of the size $n = 100, 200, 400, 800$. Use different right hand sides $f = 1, 50, 100$. Try also Gaussian function as the right hand side of (3):

$$f(x, y) = A \exp\left(-\left(\frac{(x - x_o)^2}{2\sigma_x^2} + \frac{(y - y_o)^2}{2\sigma_y^2}\right)\right), \qquad (6)$$

  for the different amplitudes A to solve (3).
  Here, x_0, y_0 is the center of the blob and σ_x, σ_y are constants which show spreading of the blob in x and y directions.

Alternative: solve the above exercise using PETSc (this will give you 5 p.).

Computer exercise 3 (1 p.)

Solve the problem of the fitting a polynomial $p(x) = \sum_{i=0}^{d} \alpha_i x^i$ of degree d to data points $(x_i, y_i), i = 1, ..., m$ in the plane by the method of normal equations and QR decomposition (Algorithm 3.1). Choose the degree of polynomial $d = 14$, the interval for $x \in [0, 1]$, discretize it by N points and compute discrete values of $y(x)$ as $y_i = y(x_i) = p(x_i)$. Our goal is to recover coefficients α_i of the polynomial $p(x) = \sum_{i=0}^{d} \alpha_i x^i$ by solving the system $A\alpha = y$ using the method of normal equations and QR decomposition (Algorithm 3.1). Here, columns of the matrix A are powers of the vector x. Compare both methods by computing the relative error e

$$e = \frac{||\alpha - \alpha^*||_2}{||\alpha^*||_2} \tag{7}$$

Here, $\alpha_i^* = 1$ are the exact values of the computed coefficients α_i.

Notes

- Compute first values of vector y_i at the points $x_i, i = 1, ..., m$ with known values of coefficients α_i. Take exact values of $\alpha_i = 1$.
- Matrix A is a Vandermonde matrix:
$$A=\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^d \\ 1 & x_2 & x_2^2 & \dots & x_2^d \\ 1 & x_3 & x_3^2 & \dots & x_3^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^d \end{bmatrix}$$
Here, $x_i, i = 1, ..., m$ are points on the interval $x \in [0, 1]$, d is degree of the polynomial.
- Use method of normal equations and QR decompisition to solve the resulting system $Ax = y$. Compare results in a table by computing the relative error (7) for both methods for different discretizations of the interval $x \in [0, 1]$.

Computer exercise 4 (1 p.)

- Solve question Q4.14 of the text book. Here we test the Matlab program eigscat.m (available to download from a course page) to plot eigenvalues of a perturbed matrix and their condition numbers.
- Solve question Q4.15 of the text book. Here we use matlab program qrplt.m (available to download from a course page) to plot the diagonal entries of a matrix undergoing unshifted QR iteration.

Computer exercise 5 (2 p.)

- Solve eigenvalue probllem for the Helmholtz equation in two dimensions
$$\triangle u(x,y) + \omega^2 u(x,y) = 0 \tag{8}$$
on the unit square $\{(x,y) : 0 < x, y < 1\}$ with boundary conditions $u = 0$ on the boundary of this square. In (8) values of ω are called eigenfrequencies.
- The equation (8) can be presented in the matrix form as
$$Au = -\omega^2 u \tag{9}$$
or with $\lambda = -\omega^2$
$$Au = \lambda u \tag{10}$$
- Write program for computing of eigenvalues and eigenvectors of (10). Using the matlab command **contour** present eigenvectors which correspond to the first 6 eigenvalues of A.
- Analyze also computed values of ω (eigenfrequencies). Use Gershgorin's theorem to do that.