Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: 2015 October 27, Time: 14.00 - 18.00, Place: at CTH, Maskinhuset

- Examiner: Larisa Beilina, tel. 070-4177036 or at work 031- 772 3567.
- Results: results of examination can be received at the latest at 10 November at the student's office at the Department of Mathematics, daily 12.30-13.00;
- Grades: to pass (get G) requires 15 points together with points from homework assignments and computer exercises.
- Solutions will be announced at the end of exam (placed on the homepage of course).
- Aids: you can use written by hand notes on the one side of A4 sheet.

Instructions

- Answer to the question carefully and clearly.
- Write on the one side of the sheet. Do not use a red pen. Do not answer more than to the one question for one page.
- Sort your answers by the order of appearance of questions. Mark on the cover answered questions. Count the number of sheets you have and fill the number of every page on the cover.

Question 1

- 1. Find eigenvalues and eigenvectors for a matrix A which is defined as

$$\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$$

  (1p)
- 3. Find inverse matrix A^{-1} to the matrix A defined as

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 5 & 0 \end{bmatrix}$$

  via the matrix of cofactors. **(2p)**
- 2. Compute $||A||_\infty, ||A||_1$ for matrix A defined as

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 5 & 0 \end{bmatrix}$$

  Find conjugate transpose matrix A^* to this matrix A. **(1p)**

Question 2

- 1. Consider LU factorization of the matrix

$$\mathbf{A} = \begin{bmatrix} 10^{-4} & 2 \\ 1 & 1 \end{bmatrix}.$$

Explain need of pivoting in LU factorization of this matrix.
(2p)
- 2. Prove that $\|Qx\|_2 = \|x\|_2$. Here, Q is orthogonal matrix $n \times n$ and x is the vector of the size n.
(0.5p)
- 3. Let the matrix H is defined as $H = I - 2\frac{vv^T}{v^Tv}$. Prove that $H^T = H$ and $H^T H = I$. Here, I is identity matrix, v is the vector and v^T is transposed vector v.
(1.5p)

Question 3

- 1. Let us consider the problem of solution of linear system of equations $Ax = b$. Let $\tilde{x}$ be approximate solution of this equation such that $\delta x = \tilde{x} - x$. Derive the upper estimate for the relative change $\frac{||\delta x||}{||\tilde{x}||}$ in terms of the condition number $k(A)$ of the matrix A and relative change of the data $\frac{||\delta A||}{||A||}$ of this matrix.
(2p)
- 2. Describe the algorithm for the improving of accuracy of a solution of linear system $Ax = b$ using Newton's method.
(2p)

Question 4

- 1. Describe three main methods (normal equations, QR decomposition and SVD) which solve the linear least squares problem. Briefly compare all these methods.
(2p)
- 2. Prove that if A is symmetric positive definite (s.p.d) matrix and H is any principal submatrix of $A(H = A(j : k, j : k)$ for some $j \leq k)$, then H is s.p.d.
(2p)
- 3. Prove that A is symmetric positive definite (s.p.d.) if and only if $A = A^T$ and all its eigenvalues are positive.
(2p)

Question 5

- Transform the given matrix A to the tridiagonal matrix using Householder transformation.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix}$$

(2p)
- Transform the given matrix A to the tridiagonal matrix using Given's rotation

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix}$$

(1p)

Question 6

- 1. Give definitions for a matrix pencil, regular and singular pencils.
  (1p)
- 2. Let $A - \lambda B$ is

$$\mathbf{A} - \lambda \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  Determine eigenvalues of $A - \lambda B$.
  (1p)

Question 7

- 1. Briefly compare Power method, Inverse iteration, Orthogonal iteration and QR iteration methods which are used to find eigenvalues and eigenvectors in the solution of non-symmetric eigenvalue problem.
  (2p)
- 2. Give definitions of upper and lower Hessenberg matrix. Illustrate Hessenberg reduction on the 4×4 matrix A= $\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$.
  (2p)

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Solutions to the examination at 27 October 2015

Question 1

1. We should solve characteristic equation $\det(A-\lambda I) = 0$:

$$\det \begin{bmatrix} 5-\lambda & 1 \\ 10 & 2-\lambda \end{bmatrix} = 0.$$

Solving above equation for λ we get eigenvalues $\lambda_1 = 7, \lambda_2 = 0$ which are solutions to the equation $\lambda(\lambda - 7) = 0$. Any vector of the form

$$x_1 = \frac{-x_2}{5}$$

will be eigenvector for the eigenvalue $\lambda = 0$. Any vector of the form

$$x_1 = \frac{x_2}{2}$$

will be eigenvector for the eigenvalue $\lambda = 7$.

2. By definition of an inverse matrix we have:

$$A^{-1} = \frac{1}{\det A}(C^T)_{ij}$$

Thus,

$$C = \begin{bmatrix} -35 & 0 & 0 \\ 25 & 0 & -5 \\ 11 & -7 & 2 \end{bmatrix},$$

$$C^T = \begin{bmatrix} -35 & 25 & 11 \\ 0 & 0 & -7 \\ 0 & -5 & 2 \end{bmatrix},$$

and thus $A^{-1} = -\frac{1}{35} \cdot \begin{bmatrix} -35 & 25 & 11 \\ 0 & 0 & -7 \\ 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -0.7143 & -0.3143 \\ 0 & 0 & 0.2 \\ 0 & 0.1429 & -0.0571 \end{bmatrix}.$

3. We use definition of A^*:

$$A^* = \overline{A^T}$$

and thus

$$A^* = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 5 \\ 5 & 7 & 0 \end{bmatrix}.$$

$||A||_1 = max(1, 10, 12) = 12$ (maximum absolute column sum), $||A||_\infty = max(9, 9, 5) = 9$ (maximum absolute row sum).

Question 2

1. See examples in Lectures 3,4 and the course book.
2. Since $Q^T Q = I$ we can write $\|Qx\|_2^2 = x^T Q^T Q x = x^T x = \|x\|_2^2$.

3. Since the matrix H is symmetric we have that $H^T = H$ and

$$H^T H = HH = \left(I - 2\frac{vv^T}{v^T v}\right)\left(I - 2\frac{vv^T}{v^T v}\right)$$
$$= I - 4\frac{vv^T}{v^T v} + 4\frac{v(v^T v)v^T}{(v^T v)(v^T v)} = I.$$

Question 3

1. See Lectures 2,3 and the course book.
   - Consider linear system $Ax = b$,
   - $\hat{x}$ such that $\hat{x} = \delta x + x$ is its computed solution.
   - Suppose $(A + \delta A)\hat{x} = b + \delta b$.
   - Goal: to bound the norm of $\delta x \equiv \hat{x} - x$.
   - Subtract the equalities and solve them for δx
   - Rearranging terms we get:

$$\delta x = A^{-1}(-\delta A \hat{x} + \delta b)$$

Taking norms and triangle inequality leads us to

$$\|\delta x\| \leq \|A^{-1}\|(\|\delta A\| \cdot \|\hat{x}\| + \|\delta b\|)$$

Rearranging inequality gives us

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq \|A^{-1}\| \cdot \|A\| \cdot (\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|A\| \cdot \|\hat{x}\|})$$

where $k(A) = \|A^{-1}\| \cdot \|A\|$ is the condition number of the matrix A

2. See Lecture 4 and the course book.

Question 4

1. See Lecture 7 and the course book.
2. See Lecture 5 and the course book proposition 2.2.
3. See Lecture 5 and the course book proposition 2.2.

Question 5

There are 2 ways how to transform matrix to the tridiagonal form with Householder transformations. We will present both of them.

Algorithm 1

- 1. We need to zero the (3, 1) entry. Apply Hausholder transformation:

  $$\mathbf{u} = \mathbf{x} + \alpha \mathbf{e}_1,$$

  where $\mathbf{x} = (0, 1)^T$, $\alpha = -sign(0) \cdot ||x||$

  $$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

  Here,

  $$\alpha = -1.$$

  Therefore

  $$\mathbf{u} = (-1, 1)^T, \quad ||u|| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

and $\mathbf{v} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$, and then

$$P' = I - 2\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the matrix of the Householder transformation is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Now, we find

$$PAP^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 4 \\ 0 & 1 & 3 \end{pmatrix}.$$

which is tridiagonal matrix.

Algorithm 2

Now let us consider the second algorithm.

– Compute α as
$\alpha = -sign(a_{21})\sqrt{a_{21}^2 + a_{31}^2} = -1$

– Compute $r = \sqrt{1/2(\alpha^2 - a_{21}\alpha)} = \frac{1}{\sqrt{2}}$

– Compute vector $v = (v_1, v_2, v_3)^T$, where $v_1 = 0, v_2 = \frac{a_{21}-\alpha}{2r} = \frac{\sqrt{2}}{2}, v_3 = \frac{a_{31}}{2r} = \frac{\sqrt{2}}{2}$.

– Compute Householder matrix $P = I - 2 \cdot v \cdot v^T$ to get

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

– Compute tridiagonal matrix as $B = PAP^T$ to get

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & 4 \\ 0 & 1 & 3 \end{pmatrix}$$

- 2. To obtain tridiagonal matrix of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix}$$

using Given's rotation we have to zero out $(3,1)$ element of the matrix A.

We construct Given's matrix G in order to zero out $(3,1)$ element of the matrix A.

To do that we compute c, s from the known $a = 0$ and $b = 1$ as

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get formulas:

$$\begin{aligned} r &= \sqrt{a^2+b^2} = \sqrt{0^2+1^2} = 1, \\ c &= \frac{a}{r} = 0, \\ s &= \frac{-b}{r} = -1. \end{aligned}$$

The Given's matrix will be

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

or

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Then tridiagonal matrix will be

$$\mathbf{G} \cdot \mathbf{A} \cdot \mathbf{G}^{\mathbf{T}} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & -4 \\ 0 & -1 & 3 \end{bmatrix}$$

Question 6

- 1. See Lecture 12 and the course book.
- 2. Characteristic polynomial will be: $p(\lambda) = \det(A - \lambda B) = (1-5\lambda)\cdot(5-\lambda)\cdot(1-0\lambda)\cdot(5-0\lambda) = (1-5\lambda)(5-\lambda) = 0$. Since $\dim p(\lambda) = 2 < 4$, then the eigenvalues are $\lambda_1 = \frac{1}{5}, \lambda_2 = 5$ and $\lambda_3 = \lambda_4 = \infty$. $\diamond$

Question 7

- 1. See Lecture 11 and the course book.
- 2. See Lecture 11 and the course book, section 4.4.6.