Efficient implementation of Helmholtz equation with applications in medical imaging

Master Project

In this project we will consider the finite element method (FEM) for the solution of Helmholtz equation

$$
\begin{aligned}
&\triangle E + \omega^2 \mu_r \varepsilon_r E = i\omega\mu_r J, \\
&\lim_{|x|\to\infty} E(x,\omega) = 0.
\end{aligned}
\tag{1}
$$

in two and three dimensions.

Solution should be implemented and tested on different real-life models in C++/PETSc in the existing software package WavES (waves24.com). The main goal of the project is efficient implementation of Helmholtz equation (1) using finite element method, and testing of the obtained solver in the already existed software package WavES. Visualization of the obtaind results will be done in Paraview/GID. It is expected that application of the obtained software will be for fast detection of small-size tumors using microwave imaging, see Fig. 1 (collaboration with Department of Electrical Engineering at CTH, Chalmers).

1 Introduction

In this project we are interested in the developing of reliable algorithms for fast implementation of Helmholtz equation (1) using finite element method.

We assume that $G \subset \mathbb{R}^n, n = 1,2,3$ is a bounded domain with a piecewise smooth boundary ∂G and $\Omega \subset G$ is another bounded domain with a boundary $\partial \Omega$. Our model problem is time-harmonic Maxwell equations in a non-magnetic, inhomogeneous and isotropic material in absence of magnetic charges, governed by the equations

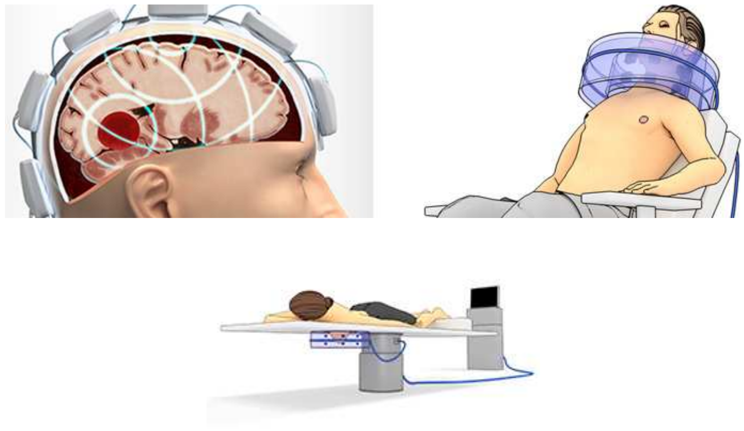

Fig. 1 Biomedical Imaging at the Department of Electrical Engineering at CTH, Chalmers. Left: setup of Stroke Finder and right: microwave hyperthermia in cancer treatment. Below: breast cancer detection using microwave tomography.

$$\nabla \times E = -\mu_r i\omega H \ \text{ in } \ \Omega, \tag{2}$$
$$\nabla \times H = \varepsilon_r i\omega E + J \ \text{ in } \ \Omega. \tag{3}$$

Here, $E = E(x,\omega)$ and $H = H(x,\omega)$ are electric and magnetic fields in frequency domain, respectively. To get (2)-(3) we applied the Fourier transform in time to the full system of Maxwell's equations such that the time-harmonic fields $A(x,\omega)$ are initialized in the form

$$A(x,\omega) = \int_0^\infty \mathbf{A}(x,t) e^{i\omega t} dt. \tag{4}$$

To solve system (2)-(3) uniquely we need Sommerfeld radiation condition at infinity

$$\lim_{|x|\to\infty} |x|^{\frac{n-1}{2}} \left(\frac{\partial}{\partial |x|} + ik \right) A = 0, \ \ n = 2,3. \tag{5}$$

where k is the wave number, see, for example, [4].

In system (2)-(3) functions ε_r and μ_r are the relative electric permittivity and the relative magnetic permeability, respectively, defined as

$$\begin{aligned} \varepsilon_r(x) &= \frac{\varepsilon(x)}{\varepsilon_r^{(0)}}, \\ \mu_r(x) &= \frac{\mu(x)}{\mu_r^{(0)}}, \end{aligned} \tag{6}$$

where ε_r^0, μ_r^0 are the dielectric permittivity and magnetic permeability of vacuum, and $\varepsilon(x), \mu(x)$ are the dielectric permittivity and magnetic permeability of the material, respectively. In (3) the function $J = J(x, \omega)$ is electric current density and is a known function.

Equations (2)-(3) are supplemented with Gauss's law

$$\begin{aligned} \nabla \cdot (\varepsilon_r E) &= 0, \\ \nabla \cdot (\mu_r H) &= 0, \end{aligned} \tag{7}$$

and perfectly conducting (PEC) boundary conditions

$$\begin{aligned} n \times E &= 0, \ \ \partial \Omega \\ n \cdot H &= 0 \ \ \partial \Omega. \end{aligned} \tag{8}$$

Here, n denotes the outer outward normal on the boundary $\partial \Omega$. We will assume that $\mu_r = const > 0.0$. Applying curl operator to the two equations of system (2)-(3) we get following vector wave equations

$$\begin{aligned} \nabla \times \nabla \times E - k^2 E &= -i\omega\mu_r J, \ x \in \Omega, \\ \nabla \times \nabla \times H - k^2 H &= \nabla \times J, \ x \in \Omega, \end{aligned} \tag{9}$$

where $k^2 = \omega^2 \mu_r \varepsilon_r$. Next, applying $\nabla \times \nabla \times f = \nabla(\nabla \cdot f) - \nabla \cdot (\nabla f)$ to the system (9) and using (8) we obtain inhomogeneous Helmholtz equations

$$\begin{aligned} \triangle E + k^2 E &= i\omega\mu_r J, \\ \triangle H + k^2 H &= -\nabla \times J. \end{aligned} \tag{10}$$

Further we will consider only the first equation of system (10) since the second one can be treated similarly.

2 Solution of the time-harmonic Maxwell's equation for electric field

We will consider the FEM numerical solution of the Helmholtz equation for the electric field in the form

$$\begin{aligned} \triangle E + \omega^2 \mu_r \varepsilon_r E &= i\omega\mu_r J, \\ \lim_{|x| \to \infty} E(x, \omega) &= 0. \end{aligned} \tag{11}$$

Solution should be implemented in C++/PETSc.

References

1. H.W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publishers, Boston, 2000.
2. Ulf Carlberg, Anders Eriksson and Per-Simon Kildal, Efficient computation of a high-q resonator realized by a thin-film multilayer metal/dielectric structure, *Microwave and Optical Technology Letters*, 44 (1), 48-51, 2005.
3. N. N. Lebedev, *Special functions and their applications*, Dover Publications, New York, 1972.
4. P. B. Monk, *Finite Element methods for Maxwell's equations*, Oxford University Press, 2003.
5. O.Pironneau, *Optimal shape design for elliptic systems*, Springer-Verlag, Berlin, 1984.
6. Z.Sipus, Per-Simon Kildal, Robert Leijon and Martin Johansson, *An algorithm for calculating Green's functions of planar, circular, cylindrical and spherical multilayer substrates*, Journal of the Applied Computational Electromagnetics Society (ACES), 13 (3), 243-254, 1998.