Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: October 25, 2016, Time: 14.00 - 18.00

- Examiner: Larisa Beilina, tel. 070-4177036 or at work 031- 772 3567.
- Results: results of examination can be received at the latest at 10 November at the student's office at the Department of Mathematics, daily 12.30-13.00;
- Grades: to pass (get G) requires 15 points together with points from homework assignments and computer exercises.
- Solutions will be announced at the end of exam (placed on the homepage of course).
- Aids: you can use written by hand notes on the one side of A4 sheet. Easy (not advanced) calculators are also allowed to use.

Instructions

- Answer to the question carefully and clearly.
- Write on the one side of the sheet. Do not use a red pen. Do not answer more than to the one question for one page.
- Sort your answers by the order of appearance of questions. Mark on the cover answered questions. Count the number of sheets you have and fill the number of every page on the cover.

Question 1

- 1. Find eigenvalues of a matrix A which is defined as

$$\mathbf{A} = \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}$$

Using information about eigenvalues of the matrix A check if the matrix A is symmetric positive definite (s.p.d.) matrix or not.
(1p)

- 2. Check if the matrix A $= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

is positive definite or not.
(2p)

3. Compute $||A||_\infty, ||A||_1$ for matrix A defined as

$$\mathbf{A} = \begin{bmatrix} -1 & 3 & -5 \\ -3 & 2 & -7 \\ -1 & 5 & 10 \end{bmatrix}$$

Find conjugate transpose matrix A^* to this matrix A. **(1p)**

Question 2

- 1. Describe procedure of partial and total pivoting in PLU factorization of the matrix $A = PLU$. Explain need of pivoting procedure.
  (2p)
- 2. Suppose that A is an invertible square matrix and u, v are vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Then the Sherman-Morrison formula states that
  $$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} uv^T A^{-1}}{1 + v^T A^{-1} u}.$$
  Here, uv^T is the outer product of two vectors u and v.
  Verify that the matrix $Y = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1} u}$ (the right-hand side of the Sherman - Morrison formula) is the inverse of a matrix $X = A + uv^T$ (the inverse of the left hand-side of the Sherman - Morrison formula) if and only if $XY = YX = \mathrm{I}$.
  (2p)

Question 3

- 1. Let us consider the problem of solution of linear system of equations $Ax = b$. Let $\tilde{x}$ be approximate solution of this equation such that $\delta x = \tilde{x} - x$. Derive the upper estimate for the relative change $\frac{||\delta x||}{||\tilde{x}||}$ in terms of the condition number $k(A)$ of the matrix A and relative change of the data $\frac{||\delta A||}{||A||}$ of this matrix.
  (2p)
- 2. Present main steps in equilibration technique to improve accuracy of a solution of linear system $Ax = b$.
  (1p)

Question 4

- 1. Let $A = QR$ be the QR decomposition of the m-by-n matrix A, where $m \geq n$. Prove that if A has a full column rank, the solution of $\min_x \|Ax - b\|_2$ is $x = R^{-1} Q^T b$.
  (2p)
- 2. Let $A = U \Sigma V^T$ be the SVD of the m-by-n matrix A, where $m \geq n$. Prove that if A has a full column rank, the solution of $\min_x \|Ax - b\|_2$ is $x = V \Sigma^{-1} U^T b$.
  (2p)

Question 5

- Transform the given matrix A to the upper Hessenberg matrix using Householder transformation.
  $$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$
  (2p)
- Transform the given matrix A to the upper Hessenberg matrix using Given's rotation
  $$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$
  (1p)

Question 6

- 1. Let $A - \lambda B$ be regular matrix pencil. Prove that if B is nonsingular, all eigenvalues of $A - \lambda B$ are finite and the same as the eigenvalues of AB^{-1} or $B^{-1}A$.
  (1p)
- 2. Let $A - \lambda B$ be regular matrix pencil. Prove that if A is nonsingular, the eigenvalues of $A - \lambda B$ are the same as the reciprocals of the eigenvalues of $A^{-1}B$ or BA^{-1}, where a zero eigenvalue of $A^{-1}B$ corresponds to an infinite eigenvalue of $A - \lambda B$.
  (2p)

Question 7

- 1. Illustrate the process of bidiagonal reduction on a 4-by-4 example with a matrix given by A= $\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$.
  (2p)
- 2. Present Jacobi method for the iterative solution of linear system of equations $Ax = b$.
  (2p)

Numerical Linear Algebra

TMA265/MMA600
Solutions to the examination at 3 January 2016

Question 1

1. We should solve characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 5-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix} = 0.$$

Solving above equation for λ we get eigenvalues $\lambda_1 = (7+\sqrt{205})/2, \lambda_2 = (7-\sqrt{205})/2$ which are solutions to the equation $\lambda^2 - 7\lambda - 39 = 0$. Since $\lambda_1 > 0$ and $\lambda_2 < 0$ then the matrix A is not symmetric positive definite.

2. The matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite since for any non-zero vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

we have

$$\begin{aligned} x^{\mathrm{T}}Ax &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} (2x_1 - x_2) & (-x_1 + 2x_2 - x_3) & (-x_2 + 2x_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= 2{x_1}^2 - 2x_1x_2 + 2{x_2}^2 - 2x_2x_3 + 2{x_3}^2 \\ &= {x_1}^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + {x_3}^2 \end{aligned}$$

which is a sum of squares and therefore nonnegative; in fact, each squared summa can be zero only when $x_1 = x_2 = x_3 = 0$, so A is indeed positive-definite.

3. We use definition of A^*:

$$A^* = \overline{A^T}$$

and thus

$$A^* = \begin{bmatrix} -1 & -3 & -1 \\ 3 & 2 & 5 \\ -5 & -7 & 10 \end{bmatrix}.$$

$||A||_1 = \max(|-1| + |-3| + |-1|, 3 + 2 + 5, |-5| + |-7| + 10) = \max(5, 10, 22) = 22$ (maximum absolute column sum),
$||A||_\infty = \max(|-1| + 3 + |-5|, |-3| + 2 + |-7|, |-1| + 5 + 10) = \max(9, 12, 16) = 16$ (maximum absolute row sum).

Question 2

1. See Lectures 3,4 and examples therein as well as the course book.
2.
We first verify that the right hand side (Y) satisfies $XY =$ I.

$$\begin{aligned} XY &= (A + uv^T)\left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}\right) \\ &= AA^{-1} + uv^TA^{-1} - \frac{AA^{-1}uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} \\ &= I + uv^TA^{-1} - \frac{uv^TA^{-1} + uv^TA^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u} \\ &= I + uv^TA^{-1} - \frac{u(1 + v^TA^{-1}u)v^TA^{-1}}{1 + v^TA^{-1}u}. \end{aligned}$$

Note that $v^TA^{-1}u$ is a scalar, so $(1 + v^TA^{-1}u)$ can be factored out, leading to:

$$XY = I + uv^TA^{-1} - uv^TA^{-1} = I.$$

In the same way, it is verified that

$$YX = \left(A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}\right)(A + uv^T) = I.$$

Question 3

1. See Lectures 2,3 and the course book.
   - Consider linear system $Ax = b$,
   - $\hat{x}$ such that $\hat{x} = \delta x + x$ is its computed solution.
   - Suppose $(A + \delta A)\hat{x} = b + \delta b$.
   - Goal: to bound the norm of $\delta x \equiv \hat{x} - x$.
   - Subtract the equalities and solve them for δx
   - Rearranging terms we get:

$$\delta x = A^{-1}(-\delta A\hat{x} + \delta b)$$

Taking norms and triangle inequality leads us to

$$\|\delta x\| \leq \|A^{-1}\|(\|\delta A\| \cdot \|\hat{x}\| + \|\delta b\|)$$

Rearranging inequality gives us

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq \|A^{-1}\| \cdot \|A\| \cdot (\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|A\| \cdot \|\hat{x}\|})$$

where $k(A) = \|A^{-1}\| \cdot \|A\|$ is the condition number of the matrix A

2. See Lecture 5 and the course book.

Equilibration technique: choose an appropriate diagonal matrix D and solve $D\mathbf{Ax} = D\mathbf{b}$ instead of $Ax = b$. D is chosen to try to make the condition number of DA smaller than that of A.

Question 4

1. See Lecture 7 (you could present any of three methods for deriving of the solution of linear least squares problem via QR decomposition) as well as the course book.
2. See Lecture 7 and the course book Theorem 3.3, part 5.

$\|Ax-b\|_2^2 = ||U\Sigma V^T x - b||_2^2$. Since A has full rank, so does Σ, and thus Σ is invertible. Now let $[U, \tilde{U}]$ be square and orthogonal as above so

$$\begin{aligned} ||U\Sigma V^T x - b||_2^2 &= \left\| \begin{bmatrix} U^T \\ \tilde{U}^T \end{bmatrix} (U\Sigma V^T x - b) \right\|_2^2 \\ &= \left\| \begin{bmatrix} \Sigma V^T x - U^T b \\ -\tilde{U}^T b \end{bmatrix} \right\|_2^2 \\ &= ||\Sigma V^T x - U^T b||_2^2 + \|\tilde{U}^T b\|_2^2. \end{aligned}$$

This is minimized by making the first term zero, i.e., $x = V\Sigma^{-1}U^T b$.

Question 5

- 1. To get upper Hessenberg matrix we need to zero the (3, 1) entry. Apply Haush-older transformation:

$$\mathbf{u} = \mathbf{x} + \alpha \mathbf{e}_1,$$

where $\mathbf{x} = (4, 3)^T$, $\alpha = -sign(4) \cdot ||x||, \|x\| = \sqrt{4^2 + 3^2} = 5, then \alpha = -5$.

We can construct $\mathbf{u} = \mathbf{x} + \alpha \mathbf{e}_1 = (4, 3)^T - 5(1, 0)^T = (-1, 3)^T$. Next, we construct

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

with $\|u\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$. Therefore $\mathbf{v} = (-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})^T$, and then

$$\begin{aligned} P' &= I - 2/10 \begin{pmatrix} -1 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \end{pmatrix} \\ &= I - 2/10 \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}. \end{aligned}$$

Then the matrix of the Householder transformation is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & 0.6 & -0.8 \end{pmatrix}$$

Now, we can get the upper Hessenberg matrix as

$$PA = \begin{pmatrix} 0 & -1 & 1 \\ 5 & 4 & 0 \\ 0 & -2 & 0 \end{pmatrix}.$$

- 2. To obtain the upper Hessenberg matrix of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

using Given's rotation we have to zero out $(3, 1)$ element of the matrix A.

We construct Given's matrix G in order to zero out $(3, 1)$ element of the matrix A.

To do that we compute c, s from the known $a = 0$ and $b = 1$ as

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get formulas:

$$r = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5,$$
$$c = \frac{a}{r} = 4/5 = 0.8,$$
$$s = \frac{-b}{r} = -3/5 = -0.6.$$

The Given's matrix will be

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

or

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{bmatrix}$$

Then the upper Hessenberg matrix will be

$$\mathbf{G} \cdot \mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 4 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Question 6

1. Proof.

If B is nonsingular and λ' is an eigenvalue, then $0 = \det(A - \lambda' B) = \det(AB^{-1} - \lambda' I) = \det(B^{-1}A - \lambda' I)$ so λ' is also an eigenvalue of AB^{-1} and $B^{-1}A$.

2. Proof.

If A is nonsingular, $\det(A - \lambda B) = 0$ and $\det(A(I - \lambda A^{-1}B) = 0$ if and only if $\det(I - \lambda A^{-1}B) = 0$ or $\det(I - \lambda BA^{-1}) = 0$. This equality can hold only if $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of $A^{-1}B$ and BA^{-1}.

Question 7

- 1. See Lecture 11 and the course book. Here is a 4-by-4 example of bidiagonal reduction, which illustrates the general pattern:

  1. Choose Q_1 so

$$Q_1 A = \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \text{ and } V_1 \text{ } so\, A_1 \equiv Q_1 A V_1 = \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}.$$

  Q_1 is a Householder reflection, and V_1 is a Householder reflection that leaves the first column of $Q_1 A$ unchanged.

  2. Choose Q_2 so

$$Q_2 A_1 = \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix} \text{ and } V_2 \text{ } so\, A_2 \equiv Q_2 A_1 V_2 = \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}.$$

Q_2 is a Householder reflection that leaves the first row of A_1 unchanged. V_2 is a Householder reflection that leaves the first two columns of Q_2A_1 unchanged.

3. Choose Q_3 so

$$Q_3A_2 = \begin{bmatrix} x & x & 0 & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix} \text{ and } V_3 = I \textit{ so } A_3 = Q_3A_2.$$

Q_3 is a Householder reflection that leaves the first two rows of A_2 unchanged. $\diamond$

In general, if A is n-by-n, then we get orthogonal matrices $Q = Q_{n-1} \cdots Q_1$ and $V = V_1 \cdots V_{n-2}$ such that $QAV = A'$ is upper bidiagonal.

Note that $A'^T A' = V^T A^T Q^T QAV = V^T A^T AV$, so $A'^T A'$ has the same eigenvalues as $A^T A$; i.e., A' has the same singular values as A.

- 2. See Lecture 11 and the course book.