Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: January 5, 2017, Time: 14.00 - 18.00

- Examiner: Larisa Beilina, tel. 070-4177036 or at work 031- 772 3567.
- Results: results of examination can be received at the student's office at the Department of Mathematics, daily 12.30-13.00;
- Grades: to pass (get G) requires 15 points together with points from homework assignments and computer exercises.
- Solutions will be announced at the end of exam (placed on the homepage of course).
- Aids: you can use written by hand notes on the one side of A4 sheet. Easy (not advanced) calculators are also allowed to use.

Instructions

- Answer to the question carefully and clearly.
- Write on the one side of the sheet. Do not use a red pen. Do not answer more than to the one question for one page.
- Sort your answers by the order of appearance of questions. Mark on the cover answered questions. Count the number of sheets you have and fill the number of every page on the cover.

Question 1

- 1. Find eigenvalues of a matrix A which is defined as

$$\mathbf{A} = \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix}$$

  Using information about eigenvalues of the matrix A check if the matrix A is symmetric positive definite (s.p.d.) matrix or not.
  (1p)
- 2. Compute condition number of the matrix A defined above in a two norm $\| \cdot \|_2$.
  (2p)
- 3. Compute $||A||_\infty, ||A||_1$ for matrix A defined as

$$\mathbf{A} = \begin{bmatrix} 100 & 300 & -50 \\ -30 & 20 & -70 \\ -100 & 50 & 10 \end{bmatrix}$$

  Find conjugate transpose matrix A^* to this matrix A. **(1p)**

Question 2

- 1. Suppose that all leading principal submatrices of matrix A of the size $n \times n$ are nonsingular. Prove that there exists a unique unit lower triangular L and nonsingular upper triangular U such that $A = LU$.
  (3p)

- 2. Suppose that A is an invertible square matrix and u, v are vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Then the Sherman-Morrison formula states that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}.$$

Here, uv^T is the outer product of two vectors u and v.

Prove that $YX = \mathrm{I}$, where $Y = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1}u}$ (the right-hand side of the Sherman - Morrison formula) and $X = A + uv^T$.

(2p)

Question 3

- 1. Let us consider the problem of solution of linear system of equations $Ax = b$. Let $\tilde{x}$ be approximate solution of this equation such that $\delta x = \tilde{x} - x$. Derive the estimate for δx using definition of the residual $r = A\tilde{x} - b$.

(2p)
- 2. Give definition of the relative condition number of the matrix A. **(1p)**

Question 4

- 1. Apply method of normal equations to solve the linear least squares problem $\min_c \|Ac - y\|_2$ of fitting to a polynomial

$$f(x, c) = \sum_{i=1}^{3} c_i x^{i-1}$$

to data points $(x_i, y_i), i = 1, ..., m$.

(2p)

Question 5

- Transform the given matrix A to the tridiagonal matrix using Householder transformation.

$$\mathbf{A} = \begin{bmatrix} 10 & 4 & 3 \\ 4 & 15 & 7 \\ 3 & 7 & 21 \end{bmatrix}$$

(2p)
- Transform the given matrix A to the tridiagonal matrix using Given's rotation

$$\mathbf{A} = \begin{bmatrix} 10 & 4 & 3 \\ 4 & 15 & 7 \\ 3 & 7 & 21 \end{bmatrix}$$

(2p)

Question 6

- 1. Briefly describe the algorithm of Tridiagonal QR iteration for the solution of symmetric eigenvalue problem.

(1p)

- 2. Give the definition of the Rayleigh quitient and discuss how and where can be applied the Rayleigh quitient.
  (2p)

Question 7

- 1. Describe the main structure of all algorithms, except Jacobi's method, to compute the SVD decomposition for a symmetric matrix A.
  (2p)
- 2. Present the Gauss-Seidel method for the iterative solution of linear system $Ax = b$.
  (2p)

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Solutions to the examination at 5 January 2017

Question 1

1. We should solve characteristic equation $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 7-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = 0.$$

Solving above equation for λ we get eigenvalues $\lambda_1 = 5 + \sqrt{5}, \lambda_2 = 5 - \sqrt{5}$ which are solutions to the equation $\lambda^2 - 10\lambda + 20 = 0$. Since $\lambda_1 > 0, \lambda_2 > 0$ then the matrix A is symmetric positive definite.

2. The condition number of the matrix A in the 2-norm is defined as $k(A) = \|A^{-1}\|_2 \cdot \|A\|_2$. The inverse of the matrix A is:

$$A^{-1} = \begin{bmatrix} 0.15 & -0.05 \\ -0.05 & 0.35. \end{bmatrix}$$

Since $\|A^{-1}\|_2 \approx 0.3618$ and $\|A\|_2 \approx 7.2361$ then $k(A) = 2.618$.

3. We use definition of A^*:

$$A^* = \overline{A^T}$$

and thus

$$A^* = \begin{bmatrix} 100 & -30 & -100 \\ 300 & 20 & 50 \\ -50 & -70 & 10 \end{bmatrix}.$$

$||A||_1 = \max(100+|-30|+|-100|, 300+20+50, |-50|+|-70|+10) = \max(230, 370, 130) = 370$ (maximum absolute column sum),
$||A||_\infty = \max(100+300+|-50|, |-30|+20+|-70|, |-100|+50+10) = \max(450, 120, 160) = 450$ (maximum absolute row sum).

Question 2

1.See Lecture 3 and the course book.

We will use induction on n. For 1-by-1 matrices we have: $a = 1 \cdot a$. To prove it for n-by-n matrices $\tilde{A}$, we need to find unique (n-1)-by-(n-1) triangular matrices L and U, unique (n-1)-by-1 vectors l and u, and unique nonzero scalar η such that

$$\tilde{A} = \begin{bmatrix} A & b \\ c^T & \delta \end{bmatrix} = \begin{bmatrix} L & 0 \\ l^T & 1 \end{bmatrix} \times \begin{bmatrix} U & u \\ 0 & \eta \end{bmatrix} = \begin{bmatrix} LU & Lu \\ l^T U & l^T u + \eta \end{bmatrix}$$

By induction unique L and U exist such that $A = LU$. Now let $u = L^{-1}b$, $l^T = c^T U^{-1}$, and $\eta = \delta - l^T u$, all of which are unique. The diagonal entries of U are nonzero by induction, and $\eta \neq 0$ since $0 \neq det\tilde{A} = det(U) \cdot \eta$.

2. See Lecture 3.

Question 3

1. See Lecture 5 and the course book.

To solve any equation $f(x) = 0$, we can try to use Newton's method to improve an approximate solution x_i to get $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$. Applying this to $f(x) = Ax - b$ yields one step of iterative refinement:

$$\begin{aligned} &r = Ax_i - b \\ &\text{solve } Ad = r \text{ for } d \\ &x_{i+1} = x_i - d \end{aligned}$$

2. See Lecture 4 and the course book. The relative condition number is given as $k_{CR}(A) = \| \, |A^{-1}| \cdot \, |A| \, \|$.

Question 4

1. See Lecture 6.

We can write the polynomial as $f(x, c) = \sum_{i=1}^{3} c_i x^{i-1} = c_1 + c_2 x + c_3 x^2$ and our data fitting problem $\min_c \|Ac - y\|$ for this polynomial takes the form

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ \vdots & \vdots & \ddots \\ 1 & x_m & x_m^2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix}.$$

The method of normal equations for $\min_c \|Ac - y\|$ will be:

$$A^T A c = A^T y,$$

then

$$c = (A^T A)^{-1} A^T y.$$

Question 5

- 1. To get tridiagonal matrix we need to zero the (3, 1) and (1,3) entries. Apply procedure of Lecture 8 of using the Householder transformation for tridiagonalization: First we compute α and r as:

$$\alpha = -\text{sgn}(a_{21}) \sqrt{\sum_{j=2}^{n} a_{j1}^2} = -\text{sgn}(4) \sqrt{\sum_{j=2}^{n} (4^2 + 3^2)} = -5;$$

$$r = \sqrt{\frac{1}{2}(\alpha^2 - a_{21}\alpha)} = 3\sqrt{\frac{5}{2}};$$

From α and r, construct vector v:

$$v^{(1)} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix},$$

where $v_1 = 0, v_2 = \frac{a_{21} - \alpha}{2r}, v_3 = \frac{a_{31}}{2r}$.

In our case we have $v^T = (0, \frac{3}{2\sqrt{\frac{5}{2}}}, \frac{1}{2\sqrt{\frac{5}{2}}})$.

Then compute:

$$P^1 = I - 2v^{(1)}(v^{(1)})^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4/5 & -3/5 \\ 0 & -3/5 & 4/5 \end{pmatrix}$$

and obtain the tridiagonal matrix $A^{(1)}$ as

$$A^{(1)} = P^1 A P^1 \approx \begin{pmatrix} 10 & -5 & 0 \\ -5 & 23.88 & -4.84 \\ 0 & -4.84 & 12.12 \end{pmatrix}$$

- 2. To obtain the tridiagonal of the matrix

$$\mathbf{A} = \begin{bmatrix} 10 & 4 & 3 \\ 4 & 15 & 7 \\ 3 & 7 & 21 \end{bmatrix}$$

using Given's rotation we have to zero out elements $(1, 3), (3, 1)$.

We construct Given's matrix G in order to zero out $(1, 3)$ element of the matrix A.

To do that we compute c, s from the known $a = 4$ and $b = 3$ as

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5, \\ c &= \frac{a}{r} = 4/5 = 0.8, \\ s &= \frac{-b}{r} = -3/5 = -0.6. \end{aligned}$$

The Given's matrix will be

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

or

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{bmatrix}$$

Then the tridiagonal matrix will be

$$\mathbf{G} \cdot \mathbf{A} \cdot \mathbf{G^T} = \begin{bmatrix} 10 & 5 & 0 \\ 5 & 23.88 & 4.84 \\ 0 & 4.84 & 12.12 \end{bmatrix}$$

Question 6

1. The algorithm of tridiagonal QR iteration for a symmetric matrix A has the following structure:

(1) Given $A = A^T$, use the variation of Algorithm 4.6 (reduction to upper Hessenberg form, see Lecture 10) to find an orthogonal Q so that $QAQ^T = T$ is tridiagonal.

(2) Apply QR iteration to T to get a sequence $T = T_0, T_1, T_2, \ldots$ of tridiagonal matrices converging to diagonal form.

2. See lecture 13.

The *Rayleigh quotient* of a symmetric matrix A and nonzero vector u is $\rho(u, A) \equiv (u^T Au)/(u^T u)$. Its largest value, $\max_{u\neq 0} \rho(u, A)$, occurs for $u = q_1$ ($\xi = e_1$) and equals $\rho(q_1, A) = \alpha_1$, the first eigenvalue of A.

Its smallest value, $\min_{u\neq 0} \rho(u, A)$, occurs for $u = q_n$ ($\xi = e_n$) and equals $\rho(q_n, A) = \alpha_n$, the last eigenvalue of A.

Question 7

- 1. See Lecture 14 and the course book.

  All the algorithms for the SVD of a general matrix G, except Jacobi's method, have an analogous structure:
  (1) Reduce G to bidiagonal form B with orthogonal matrices U_1 and V_1: $G = U_1 B V_1^T$. This means B is nonzero only on the main diagonal and first superdiagonal.
  (2) Find the SVD of B: $B = U_2 \Sigma V_2^T$, where Σ is the diagonal matrix of singular values, and U_2 and V_2 are orthogonal matrices whose columns are the left and right singular vectors, respectively.
  (3) Combine these decompositions to get $G = (U_1 U_2)\Sigma(V_1 V_2)^T$. The columns of $U = U_1 U_2$ and $V = V_1 V_2$ are the left and right singular vectors of G, respectively.
- 2. See Lecture 11 and the course book. To get the Gauss-Seidel method we use the same splitting as for the Jacobi method. Applying it to the solution of $Ax = b$ we have:

$$Ax = Dx - (\tilde{L}x + \tilde{U}x) = b,$$

we rearrange terms in the right hand side now like that:

$$Dx - \tilde{L}x = b + \tilde{U}x \tag{0.1}$$

and thus now the solution is computed as

$$x = (D - \tilde{L})^{-1}(b + \tilde{U}x) = (D - \tilde{L})^{-1}b + (D - \tilde{L})^{-1}\tilde{U}x.$$

We can rewrite the above equation using notations $DL = \tilde{L}$ and $DU = \tilde{U}$ as:

$$\begin{aligned} x &= (D - \tilde{L})^{-1}b + (D - \tilde{L})^{-1}\tilde{U}x \\ &= (D - DL)^{-1}b + (D - DL)^{-1}\tilde{U}x \\ &= (I - L)^{-1}D^{-1}b + (I - L)^{-1}D^{-1}\tilde{U}x \\ &= (I - L)^{-1}D^{-1}b + (I - L)^{-1}Ux. \end{aligned} \tag{0.2}$$

Let us define

$$\begin{aligned} R_{GS} &\equiv (I - L)^{-1}U, \\ c_{GS} &\equiv (I - L)^{-1}D^{-1}b. \end{aligned} \tag{0.3}$$

Then iterative update in the Gauss-Seidel method can be written as:

$$x_{m+1} = R_{GS}x_m + c_{GS}. \tag{0.4}$$