Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: October 24, 2017, Time: 14.00 - 18.00

Question 1

- 1. Using definition of singular values σ, find singular values of a matrix A which is defined as

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 5 \end{bmatrix}$$

Compute $\|A\|_2$ of a matrix A.

(2p)

- 2. Find conjugate transpose matrix A^* of the following matrix

$$\mathbf{A} = \begin{bmatrix} 15+2i & 21-i & 5i \\ 1-3i & i & -5-i \end{bmatrix}$$

(0.5p)

- 3. Compute $||A||_\infty, ||A||_1, \|A\|_2$ for a matrix A defined as

$$\mathbf{A} = \begin{bmatrix} 7 & -10 & 0 \\ -10 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

(2p)

Question 2

- 1. Describe procedure of partial and total pivoting in PLU factorization of the matrix $A = PLU$. Explain need of pivoting procedure.

(1.5p)

- 2. Let a matrix A is of the size $n \times n$ and be a symmetric, positive definite matrix. Let $x \in R^n$ be a vector. Using the above information derive procedure how to compute $x^T A^{-1} x$.

(2p)

Question 3

- 1. Let us consider the problem of solution of linear system of equations $Ax = b$. Let $\tilde{x}$ be approximate solution of this equation such that $\delta x = \tilde{x} - x$. Derive the upper estimate for the relative change $\frac{||\delta x||}{||\tilde{x}||}$ in terms of the condition number $k(A)$ of the matrix A and relative change of the data $\frac{||\delta A||}{||A||}$ of this matrix.

(2p)

- 2. Present algorithm for the iterative refinement technique to improve accuracy of a solution of linear system $Ax = b$.

(2p)

Question 4

- 1. Let A has full rank and $A = QR$ be the QR decomposition of the m-by-n matrix A, where $m \geq n$. Derive formula for the solution of the least squares problem $\min_x \|Ax - b\|_2^2$ using the QR factorization of a matrix A.
  (2p)
- 2. Let A has full rank and $A = U\Sigma V^T$ be the SVD of the m-by-n matrix A, where $m \geq n$. Derive formula for the solution of the least squares problem $\min_x \|Ax - b\|_2^2$ using the SVD decomposition of a matrix A.
  (2p)

Question 5

- Transform the given matrix A to the lower Hessenberg matrix using Householder transformation.

$$\mathbf{A} = \begin{bmatrix} 0 & -4 & 3 \\ 3 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

  (2p)
- Transform the given matrix A to the lower Hessenberg matrix using Given's rotation

$$\mathbf{A} = \begin{bmatrix} 0 & -4 & 3 \\ 3 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

  (1p)

Question 6

- 1. State and prove the Gerschgorin's theorem.
  (2p)
- 2. Give definition of the real Schur canonical form.
  (1p)

Question 7

- 1. Let a matrix A be of the size $m \times n$ and a matrix B be of the size $n \times m$. Show that the block matrices

$$\mathbf{X} = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$$

  and

$$\mathbf{Y} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

  are similar.
  (2p)
- 2. Briefly describe the Power method for the solution of the non-symmetric eigenproblem.
  (1p)

Numerical Linear Algebra

TMA265/MMA600
Solutions to the examination at 24 October 2017

Question 1

1. By definition of singular values $\sigma = \sqrt{\lambda(A^*A)}$ we have:

$$A^* = \overline{A^T} = \begin{bmatrix} 3 & 1 \\ 0 & 5 \end{bmatrix}, \quad A^*A = \begin{bmatrix} 10 & 5 \\ 5 & 25 \end{bmatrix},$$

and characteristic equation to solve for λ is $\lambda^2 - 35\lambda + 225 = 0$.

Solving above equation we get eigenvalues $\lambda_1 = 8.4861, \lambda_2 = 26.5139$. Then singular values will be: $\sigma_1 = \sqrt{\lambda_1} = 2.9131, \sigma_2 = \sqrt{\lambda_2} = 5.1492$, and $\|A\|_2 = \max(\sigma_1, \sigma_2) = 5.1492$.

2. The conjugate transpose matrix will be:

$$A^* = \begin{bmatrix} 15 - 2i & 1 + 3i \\ 21 + i & -i \\ -5i & -5 + i \end{bmatrix}$$

3. To compute $\|A\|_2$ we use definition of A^*:

$$A^* = \overline{A^T}$$

and thus

$$A^* = \begin{bmatrix} 7 & -10 & 0 \\ -10 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Then

$$A^*A = \begin{bmatrix} 149 & -120 & -10 \\ -120 & 126 & 8 \\ -10 & 8 & 10 \end{bmatrix}$$

and $\lambda(A^*A) = (9.2592, 17.0342, 258.7066)$, $\sqrt{\lambda(A^*A)} = (3.0429, 4.1273, 16.0844)$, then $\|A\|_2 = \max(3.0429, 4.1273, 16.0844) = 16.0844$.

$||A||_1 = \max(|7| + |-10| + 0, |-10| + 5 + 1, 0 + 1 + 3) = \max(17, 16, 4) = 17$ (maximum absolute column sum),
$||A||_\infty = 17$ (maximum absolute row sum).

Question 2

1. See Lectures 3,4 and examples therein as well as the course book.

2. We can use Cholesky decomposition $A = LL^T$ of a matrix A. Then $x^TAx = x^TLL^Tx$, and $x^TA^{-1}x = x^T(LL^T)^{-1}x = x^TL^{-T}L^{-1}x$. Supposing that $Ly = x$ can be solved by forward substitution we have that $y = L^{-1}x$ and $y^T = x^TL^{-T}$ and thus $y^Ty = x^TA^{-1}x$.

Question 3

1. See Lecture 3.

2. See Lecture 3 and the course book. Applying Newton's method to $f(x) = Ax - b$ yields one step of iterative refinement:

$$\begin{aligned}&r = Ax_i - b\\ &\text{solve } Ad = r \text{ for } d\\ &x_{i+1} = x_i - d\end{aligned}$$

Question 4

1. Take any of 3 different proofs in Lecture 7.
2. See Lecture 7:

If A has full rank, the solution of $\min_x \|Ax - b\|_2$ is $x = V\Sigma^{-1}U^Tb$.

$\|Ax - b\|_2^2 = ||U\Sigma V^Tx - b||_2^2$. Since A has full rank, so does Σ, and thus Σ is invertible. Now let $[U, \tilde{U}]$ be square and orthogonal as above so

$$\begin{aligned} ||U\Sigma V^Tx - b||_2^2 &= \left\| \begin{bmatrix} U^T \\ \tilde{U}^T \end{bmatrix} (U\Sigma V^Tx - b) \right\|_2^2 \\ &= \left\| \begin{bmatrix} \Sigma V^Tx - U^Tb \\ -\tilde{U}^Tb \end{bmatrix} \right\|_2^2 \\ &= ||\Sigma V^Tx - U^Tb||_2^2 + \|\tilde{U}^Tb\|_2^2. \end{aligned}$$

This is minimized by making the first term zero, i.e., $x = V\Sigma^{-1}U^Tb$.

Question 5

- 1. To get lower Hessenberg matrix we need to zero the (1, 3) entry of A.

  Apply Hausholder transformation to A^T:

$$\mathbf{A}^T = \begin{bmatrix} 0 & 3 & 4 \\ -4 & 4 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

  We have:

$$\mathbf{u} = \mathbf{x} + \alpha\mathbf{e}_1,$$

where $\mathbf{x} = (-4, 3)^T$, $\alpha = -sign(-4) \cdot ||x||$, $\|x\| = \sqrt{(-4)^2 + 3^2} = 5$, then $\alpha = 5$.

  We can construct $\mathbf{u} = \mathbf{x} + \alpha\mathbf{e}_1 = (-4, 3)^T + 5(1, 0)^T = (1, 3)^T$. Next, we construct

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

with $\|u\| = \sqrt{(1)^2 + 3^2} = \sqrt{10}$. Therefore $\mathbf{v} = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})^T$, and then

$$\begin{aligned} P' &= I - 2/10 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} \\ &= I - 2/10 \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 4/5 & -3/5 \\ -3/5 & -4/5 \end{pmatrix}. \end{aligned}$$

Then the matrix of the Householder transformation is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8 & -0.6 \\ 0 & -0.6 & -0.8 \end{pmatrix}$$

Now, we can get the lower Hessenberg matrix as

$$AP = \begin{pmatrix} 0 & -5 & 0 \\ 3 & 3.2 & -2.4 \\ 4 & -1.2 & -1.6 \end{pmatrix}.$$

- 2. To obtain the lower Hessenberg matrix of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -4 & 3 \\ 3 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

using Given's rotation we have to zero out $(1,3)$ element of the matrix A.

Let's consider A^T of A:

$$\mathbf{A}^T = \begin{bmatrix} 0 & 3 & 4 \\ -4 & 4 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

We construct Given's matrix G in order to zero out $(3,1)$ element of the matrix A^T.

To do that we compute c, s from the known $a = -4$ and $b = 3$ as

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get formulas:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} = \sqrt{(-4)^2 + 3^2} = 5, \\ c &= \frac{a}{r} = -4/5, \\ s &= \frac{-b}{r} = 3/5. \end{aligned}$$

The Given's matrix will be

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

or

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.8 & 0.6 \\ 0 & -0.6 & -0.8 \end{bmatrix}$$

Then the lower Hessenberg matrix will be

$$\mathbf{A} \cdot \mathbf{G^T} = \begin{bmatrix} 0 & 5 & 0 \\ 3 & -3.2 & -2.4 \\ 4 & 1.2 & -1.6 \end{bmatrix}$$

The Givens matrix can be also constructed as

$$\mathbf{G} = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}$$

for $a = 0, b = 3$ and thus $r = \sqrt{0^2 + 3^2} = 3, c = 0, s = -1$:

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Then the lower Hessenberg matrix will be

$$\mathbf{AG^T} = \begin{bmatrix} 3 & -4 & 0 \\ 0 & 4 & -3 \\ 2 & 0 & -4 \end{bmatrix}$$

Question 6

- 1. Gershgorin's theorem: *Let B be an arbitrary matrix. Then the eigenvalues λ of B are located in the union of the n disks*

$$|\lambda - b_{kk}| \leq \sum_{j \neq k} |b_{kj}|.$$

Proof. Given λ and $x \neq 0$ such that $Bx = \lambda x$, let $1 = ||x||_\infty = x_k$ by scaling x if necessary. Then $\sum_{j=1}^N b_{kj} x_j = \lambda x_k = \lambda$, so $\lambda - b_{kk} = \sum_{\substack{j=1 \\ j \neq k}}^N b_{kj} x_j$, implying

$$|\lambda - b_{kk}| \leq \sum_{j \neq k} |b_{kj} x_j| \leq |b_{kj}|. \quad \square$$

- 2. *Real Schur canonical form.* If A is real, there exists a real orthogonal matrix V such that $V^T AV = T$ is quasi-upper triangular. This means that T is block upper triangular with 1-by-1 and 2-by-2 blocks on the diagonal. Its eigenvalues are the eigenvalues of its diagonal blocks. The 1-by-1 blocks correspond to real eigenvalues, and the 2-by-2 blocks to complex conjugate pairs of eigenvalues.

Question 7

- 1. Let $Y = S^{-1}XS$, so X and Y are similar. By properties of similar matrices, X and Y have the same eigenvalues. In our case we have that eigenvalues of X are the same as eigenvalues of Y.

$$\det(X - \lambda I) = \det(AB - \lambda I) \cdot (-\lambda I) = 0,$$

$$\det(Y - \lambda I) = \det(-\lambda I)(BA - \lambda I) = 0.$$

- 2. Power method: Given x_0, we iterate

$$
\begin{array}{ll}
i = 0 & \\
\textit{repeat} & \\
\quad y_{i+1} = Ax_i & \\
\quad x_{i+1} = y_{i+1}/||y_{i+1}||_2 & (\textit{approximate eigenvector}) \\
\quad \tilde{\lambda}_{i+1} = x_{i+1}^T A x_{i+1} & (\textit{approximate eigenvalue}) \\
\quad i = i + 1 & \\
\textit{until convergence} &
\end{array}
$$