Questions for the course

Numerical Linear Algebra

TMA265/MMA600

Date: January 5, 2018

Question 1

- 1. Using definition of singular values σ, find singular values of a matrix A which is defined as
$$\mathbf{A} = \begin{bmatrix} 1-i & 0 \\ 1 & 2+i \end{bmatrix}$$
Compute the spectral norm $\|A\|_2$ of a matrix A.
(2p)
- 2. Give definition of a symmetric, indefinite matrix.
(1p)
- 3. Give definition of the nonlinear eigenvalue problem which is also called matrix polynomial.
(1p)

Question 2

- 1. Explain need of pivoting procedure by doing LU decomposition of a matrix
$$\mathbf{A} = \begin{bmatrix} 10^{-2} & 1 \\ 1 & 1 \end{bmatrix}$$
(2p)
- 2.
Let $H = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$, where A is square and $A = U\Sigma V^T$ is the SVD of A. Let $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$, $U = [u_1, \ldots, u_n]$, and $V = [v_1, \ldots, v_n]$. Prove that then the $2n$ eigenvalues of H are $\pm\sigma_i$, and corresponding eigenvectors are $\frac{1}{\sqrt{2}} \begin{bmatrix} v_i \\ \pm u_i \end{bmatrix}$.
(3p)

Question 3

- 1. Derive stability result for the solution of the problem $Ax = b$.
(2p)
- 2. Prove that pseudoinverse $A^+$of a matrix A of order $m \times n$ satisfies $AA^+A = A$.
(2p)

Question 4

- Suppose that A is m by n with $m > n$ and is rank-deficient with rank $r < n$. Using the SVD decomposition of A give definition of the Moore-Penrose pseudoinverse A^+ for rank-deficient A.
(2p)

- Derive formula for the solution of least squares problem $\min_x \|Ax - b\|_2^2$ using the method of normal equations.
  (2p)

Question 5

- Transform the given matrix A to the upper Hessenberg matrix using Householder transformation.

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

  (2p)
- Transform the given matrix A to the upper Hessenberg matrix using Given's rotation

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

  (1p)

Question 6

- 1. Let Q is an orthogonal matrix, $\dim Q = n \times n$. Prove that $\|Qx\|_2 = \|x\|_2$.
  (1p)
- 2. Let $Q^* AQ = T$ be the Schur canonical form. Describe how to compute eigenvectors of A by knowing the Schur canonical form T.
  (2p)

Question 7

- 1. Briefly present algorithm of QR iteration for the solution of the non-symmetric eigenproblems. **(2p)**
- 2. Describe procedure how to choose *Wilkonson's shift* in the algorithm of QR iteration with shift which is used for the solution of the non-symmetric eigenproblems.
  (2p)

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Solutions to the examination at 5 January 2018

Question 1

1. By definition of singular values $\sigma = \sqrt{\lambda(A^*A)}$ we have:

$$A^* = \overline{A^T} = \begin{bmatrix} 1+i & 1 \\ 0 & 2-i \end{bmatrix}, \quad A^*A = \begin{bmatrix} 3 & 2+i \\ 2-i & 5 \end{bmatrix},$$

and characteristic equation to solve for λ is $\lambda^2 - 8\lambda + 10 = 0$.

Solving above equation we get eigenvalues $\lambda_{1,2} = 4 \pm \sqrt{6}$, or $\lambda_1 = 6.4495, \lambda_2 = 1.5505$. Then singular values will be: $\sigma_1 = \sqrt{\lambda_1} = 2.5396, \sigma_2 = \sqrt{\lambda_2} = 1.2452$, and $\|A\|_2 = \max(\sigma_1, \sigma_2) = 2.5396$.

2. Symmetric: $A = A^T$, Indefinite matrix: if for some vectors x we have $x^T Ax > 0$ and for some other vectors x we have $x^T Ax < 0$. We can also define an indefinite matrix as a Hermitian matrix which is neither positive definite, negative definite, positive semidefinite, nor negative semidefinite.

3. The *nonlinear eigenvalue problem* or *matrix polynomial* is defined as

$$\sum_{i=0}^{d} \lambda^i A_i = \lambda^d A_d + \lambda^{d-1} A_{d-1} + \cdots + \lambda A_1 + A_0.$$

Question 2

1. See Lectures 3,4 and examples therein as well as the course book.
LU decomposition without pivoting gives following L and U:

$$L = \begin{bmatrix} 1 & 0 \\ 10^2 & 1 \end{bmatrix}, U = \begin{bmatrix} 10^{-2} & 1 \\ 0 & -10^2 \end{bmatrix}$$

Then LU is not the same as A- we loss accuracy without pivoting :

$$LU = \begin{bmatrix} 10^{-2} & 1 \\ 1 & 0 \end{bmatrix}$$

Compare the condition number of A to the condition numbers of L and U. In our case:

- $k(A) = ||A||_\infty \cdot ||A^{-1}||_\infty \approx 4$
- $k(L) = ||L||_\infty \cdot ||L^{-1}||_\infty \approx 10^4$
- $k(U) = ||U||_\infty \cdot ||U^{-1}||_\infty \approx 10^4$

Since $k(A) << k(L) \cdot k(U)$ - warning:loss of accuracy.

2.
We substitute $A = U\Sigma V^T$ into H to get: $H = \begin{bmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{bmatrix}$

Choose orthogonal matrix G such that

$$G = \frac{1}{\sqrt{2}} \begin{bmatrix} V & V \\ U & -U \end{bmatrix}$$

It is orthogonal since $I = GG^T = \frac{1}{2} \begin{bmatrix} VV^T + VV^T & 0 \\ 0 & UU^T + UU^T \end{bmatrix}$

Then we observe that

$$G \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix} G^T = \begin{bmatrix} 0 & V\Sigma U^T \\ U\Sigma V^T & 0 \end{bmatrix} = H$$

Then using the spectral theorem we can conclude that the $2n$ eigenvalues of H are $\pm\sigma_i$, with corresponding eigenvectors $\frac{1}{\sqrt{2}} \begin{bmatrix} v_i \\ \pm u_i \end{bmatrix}$.

Question 3

1. We have: $Ax = b$, $A(x + \delta x) = b + \delta b$, $\delta x = A^{-1}\delta b$. Take norms: $\|\delta x\| \leq \|A^{-1}\|\|\delta b\|$. Then $\frac{\|\delta x\|}{\|x\|} \leq \|A^{-1}\|\|A\|\frac{\|\delta b\|}{\|A\|\|x\|} = k(A)\frac{\|\delta b\|}{\|A\|\|x\|} \leq k(A)\frac{\|\delta b\|}{\|b\|}$

2. To prove that $AA^+A = A$ we use definition of $A^+ = (A^TA)^{-1}A^T$:

$$AA^+A = A[(A^TA)^{-1}A^T]A = A\underbrace{(A^TA)^{-1}A^TA}_{I} = A.$$

Question 4

1. Suppose that A is m by n with $m > n$ and is rank-deficient with rank $r < n$. Let $A = U\Sigma V^T = U_1\Sigma_1 V_1^T$ being a SVD decompositions of A such that

A $=[\mathrm{U}_1, U_2] \left[\begin{array}{c|c} \Sigma_1 & 0 \\ \hline 0 & 0 \end{array}\right] [V_1, V_2]^T = U_1\Sigma_1 V_1^T$

Here, $size(\Sigma_1) = r \times r$ and is nonsingular, U_1 and V_1 have r columns. Then

$$A^+ \equiv V_1\Sigma_1^{-1}U_1^T$$

is called the Moore-Penrose pseudoinverse for rank-deficient A.

2. See Lecture 7:

$$\begin{aligned} 0 &= \lim_{e\to 0}\frac{(A(x+e)-b)^T(A(x+e)-b)-(Ax-b)^T(Ax-b)}{||e||_2} \\ &= \lim_{e\to 0}\frac{2e^T(A^TAx-A^Tb)+e^TA^TAe}{||e||_2} \end{aligned}$$

The second term $\frac{|e^TA^TAe|}{||e||_2} \leq \frac{||A||_2^2||e||_2^2}{||e||_2} = ||A||_2^2||e||_2^2$ approaches 0 as e goes to 0, so the factor $A^TAx - A^Tb$ in the first term must also be zero, or $A^TAx = A^Tb$. This is a system of n linear equations in n unknowns, the normal equations.

Question 5

- 1. To get upper Hessenberg matrix, we need to zero the $(3,1)$ entry of A.
  We have:

$$\mathbf{u} = \mathbf{x} + \alpha \mathbf{e}_1,$$

where $\mathbf{x} = (3,4)^T$, $\alpha = -sign(3) \cdot ||x||$, $\|x\| = \sqrt{(4)^2 + 3^2} = 5$, then $\alpha = -5$.
  We can construct $\mathbf{u} = \mathbf{x} + \alpha \mathbf{e}_1 = (3,4)^T - 5(1,0)^T = (-2,4)^T$. Next, we construct

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

with $\|u\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$. Therefore $\mathbf{v} = (\frac{-2}{\sqrt{20}}, \frac{4}{\sqrt{20}})^T$, and then

$$\begin{aligned} P' &= I - 2/20 \begin{pmatrix} -2 \\ 4 \end{pmatrix} \begin{pmatrix} -2 & 4 \end{pmatrix} \\ &= I - 2/20 \begin{pmatrix} 4 & -8 \\ -8 & 16 \end{pmatrix} \\ &= \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}. \end{aligned}$$

Then the matrix of the Householder transformation is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & 0.8 & -0.6 \end{pmatrix}$$

  Now, we can get the upper Hessenberg matrix as

$$PA = \begin{pmatrix} 0 & -1 & 1 \\ 5 & 2.4 & 1.6 \\ 0 & 3.2 & -1.2 \end{pmatrix}.$$

- 2. To obtain the upper Hessenberg matrix of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

using Given's rotation we have to zero out $(3,1)$ element of the matrix A.
  To do that we compute c, s from the known $a = 3$ and $b = 4$ as

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

to get formulas:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} = \sqrt{(3)^2 + 4^2} = 5, \\ c &= \frac{a}{r} = 3/5, \\ s &= \frac{-b}{r} = -4/5. \end{aligned}$$

The Given's matrix will be

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

or

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix}$$

Then the upper Hessenberg matrix will be

$$\mathbf{GA} = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 2.4 & 1.6 \\ 0 & -3.2 & 1.2 \end{bmatrix}$$

The Givens matrix can be also constructed as

$$\mathbf{G} = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}$$

for $a = 0, b = 4$ and thus $r = \sqrt{0^2 + 4^2} = 4, c = 0, s = -1$:

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Then the upper Hessenberg matrix will be

$$\mathbf{GA} = \begin{bmatrix} 4 & 0 & 2 \\ 3 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Question 6

1. $\|Qx\|_2^2 = x^TQ^TQx = x^Tx = \|x\|_2^2$ and thus $\|Qx\|_2 = \|x\|_2$.

2. Let $Q^*AQ = T$ be the Schur form. Then if $Tx = \lambda x$, we have

$$Tx = \lambda x \rightarrow (Q^*AQ)x = Tx = \lambda x \rightarrow AQx = QTx = \lambda Qx,$$

and Qx is an eigenvector of A. Thus, to find eigenvectors Qx of A, it suffices to find eigenvectors x of T.

Question 7

1. Algorithm of QR iteration: Given A_0, we iterate

$$
\begin{array}{l}
i = 0 \\
\textit{repeat} \\
\quad Factorize, A_i = Q_iR_i \quad (\textit{the QR decomposition}) \\
\quad A_{i+1} = R_iQ_i \\
\quad i = i + 1 \\
\textit{until convergence}
\end{array}
$$

2. How to choose *Wilkonson's shift*: let shift σ_i is chosen as an eigenvalue of the matrix

$$\begin{bmatrix} a_{n-1,n-1} & a_{n-1,n} \\ a_{n,n-1} & a_{n,n} \end{bmatrix}$$

which is closest to the value $a_{n,n}$ of the matrix A_i.