Perturbation theory

(1) $A x = b$

(2) $(A+\delta A)(\underbrace{x+\delta x}_{\hat{x}}) = b + \delta b$

(2) − (1): $\cancel{Ax} + A\delta x + \delta A\, x + \delta A \cdot \delta x - \cancel{Ax} = \cancel{b} + \delta b - \cancel{b}$

$$A\delta x + \delta A\, x + \delta A\, \delta x = \delta b$$

$$(A + \delta A)\,\delta x + \delta A\, x = \delta b$$

$$(A+\delta A)\,\delta x = \delta b - \delta A \cdot x =$$

$$= \delta b - \delta A(\hat{x} - \delta x)$$

$$= \delta b - \delta A \cdot \hat{x} + \delta A \cdot \delta x$$

$$A\delta x + \cancel{\delta A \delta x} = \delta b - \delta A \cdot \hat{x} + \cancel{\delta A \cdot \delta x}$$

$$\boxed{\delta x = A^{-1}\left[\delta b - \delta A \cdot \hat{x}\right]}$$

If $r = A\hat{x} - b$ then

$(A + \delta A)\hat{x} = b$ if $\delta A \cdot \hat{x} = -r$.

$\Updownarrow$

$A\hat{x} + \delta A \cdot \hat{x} = b$

$\downarrow$

$\delta A \cdot \hat{x} = b - A \cdot \hat{x} = -r$

$\downarrow$

$\delta A \cdot \hat{x} = -r \Rightarrow \|r\| = \|\delta A \cdot \hat{x}\| \leq$

$\leq \|\delta A\| \cdot \|\hat{x}\| \Rightarrow$

$$\Rightarrow \|\delta A\| \geq \frac{\|r\|}{\|\hat{x}\|}$$

$\Leftarrow$ Notes for Lecture 3, Theorem after "Practical error bou[nd]"

Practical error bounds

(1) $A x = b$

(2) $\tilde{x} = x + \delta x$ → computed solution of (1)

(3) $r = A \cdot \tilde{x} - b$ – residual

Error $\delta x = \tilde{x} - x = \tilde{x} - \overbrace{A^{-1} b}^{\text{from (1)}} =$

$= \underbrace{A^{-1}(r+b)}_{\text{from (3)}} - A^{-1} b =$

$= A^{-1} r + A^{-1} b - A^{-1} b =$

$= A^{-1} r$

$$\|\delta x\| \leq \|A^{-1}\| \cdot \|r\|$$

Leading submatrix

$= A(1:1; 1:1)$

$$A = n\left\{\overbrace{\begin{bmatrix} \boxed{a_{11}} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}^{n}\right.$$

$A(1:j;1:j)$ - leading submatrix av A

Om $\det(A(1:j;1:j)) \neq 0$, då existerar $A = LU$.

Exempel:

$$A = \begin{bmatrix} 2 & 6 \\ 4 & 15 \end{bmatrix}$$

Kollar: $\det(A(1:1;1:1)) = 2 > 0$

$\det(A(1:2;1:2)) = \det\left(\begin{bmatrix} 2 & 6 \\ 4 & 15 \end{bmatrix}\right) =$

$= 2 \cdot 15 - 24 = 6 > 0$

Vi kan göra $A = LU$ faktorisering.

Inversion of 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} := \frac{1}{\det A} \cdot C^{T}$$

C for A? $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$C_{11} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{1+1} \cdot d = d$$

$$C_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{1+2} \cdot c = -c$$

$$C_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{2+1} \cdot b = -b$$

$$C_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{2+2} \cdot a = a$$

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \Rightarrow C^{T} = \begin{bmatrix} d; & -b \\ -c; & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} d; & -b \\ -c; & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d; & -b \\ -c; & a \end{bmatrix}$$

Inversion of 3x3 matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C = \begin{bmatrix} ek-fh; & gf-dk; & dh-ge \\ hc-bk; & ak-gc; & gb-ah \\ bf-ec; & dc-af; & ae-db \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$$C_{11} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} e & f \\ h & k \end{vmatrix} = ek - fh$$

$$C_{12} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{12} = (-1)^{1+2} \begin{vmatrix} d & f \\ g & k \end{vmatrix} = -(dk - gf)$$

$$C_{13} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{13} = (-1)^{1+3} \begin{vmatrix} d & e \\ g & h \end{vmatrix} = dh - ge$$

$$C_{21} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{21} = (-1)^{2+1} \begin{vmatrix} b & c \\ h & k \end{vmatrix} = -(bk - hc)$$

$$C_{22} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{22} = (-1)^{2+2} \begin{vmatrix} a & c \\ g & k \end{vmatrix} = ak - gc$$

$$C_{23} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{23} = (-1)^{2+3} \begin{vmatrix} a & b \\ g & h \end{vmatrix} = -(ah - gb)$$

$$C_{31} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{31} = (-1)^{3+1} \cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix} = bf - ec$$

$$C_{32} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} a & c \\ d & f \end{vmatrix} = -(af - dc)$$

$$C_{33} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{33} = (-1)^{3+3} \cdot \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - db$$

Gaussian elimination with partial pivoting

rhs

find pivot element in the 1-th column

$$\left[\begin{array}{cccc|c} 0.02 & 0.01 & 0 & 0 & 0.02 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

A

To zero out a_{21} compute $m_{21} = \frac{a_{21}}{a_{11}} = \frac{0.02}{1} = 0.02$

$E_2 - m_{21}E_1$

$$\left[\begin{array}{cccc|c} \boxed{1 & 2 & 1 & 0 & 1} \\ 0.02 & 0.01 & 0 & 0 & 0.02 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

← fix this line and perform 1 Step of Gauss. elimination

element

zero out

Perform now Gaussian elimination: $m_{21} = \frac{a_{21}}{a_{11}}$

1 step

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

→ fixed line

$(E_2 - m_{21}E_1) \to$

find pivot element

$\Downarrow$

already fixed line

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

→ fix now this line

1) $m_{32} = \frac{a_{32}}{a_{22}} = \frac{-0.03}{1} = -0.03$

Perform 1 Step of Gaussian elimination to zero out -0.03

2) $E_3 - m_{32}E_2$

$= [0; -0.03; -0.02; 0 \| 0]$

$- [-0.03 \cdot 1; -0.03 \cdot 2; -0.03 \cdot 1 \| -0.03 \cdot 4]$

$= [0; 0; 0.04; 0.03 \| 0.12]$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0.04 & 0.03 & 0.12 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right] \begin{array}{l} \rightarrow \text{fixed} \\ \rightarrow \text{fixed} \\ \\ \\ \end{array}$$

find pivot element in the 3-rd column

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \\ 0 & 0 & 0.04 & 0.03 & 0.12 \end{array}\right] \begin{array}{l} \rightarrow \text{fixed} \\ \rightarrow \text{fixed} \\ \\ \\ \end{array}$$

Perform 1 step of Gauss elim. to zero out 0.04: 1) $m_{43} = \frac{0.04}{100} =$

2) $E_4 - m_{43} \cdot E_3 =$

$= [0; 0; 0.04; 0.03 | 0.12] - [0; 0; \frac{0.04}{100} \cdot 100; \frac{0.04}{100} \cdot 200 | \frac{0.04}{100} \cdot 800]$

$$\Downarrow$$

$$\underbrace{\left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 100 & 200 \\ 0 & 0 & 0 & -0.05 \end{array}\right.}_{A} \cdot x \quad \underbrace{\left|\begin{array}{c} 1 \\ 4 \\ 800 \\ -0.2 \end{array}\right]}_{b}$$

Obtain solution by backward substitution:

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$-0.05\, x_4 = -0.2 \Rightarrow x_4 = 4$

$100 x_3 + 200 x_4 = 800 \Rightarrow x_3 = 0$

$x_2 + 2 x_3 + x_4 = 4 \Rightarrow x_2 = 0$

$x_1 + 2 x_2 + x_3 + 1 \Rightarrow x_1 = 0$

Ex. 1 Need of Pivoting

$$A = \begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix}$$

Compute $\kappa(A) = \|A^{-1}\|_\infty \cdot \|A\|_\infty$

Compute A^{-1} : $A^{-1} = \frac{1}{\det A} C^T$

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 10^{-4} \end{bmatrix} ; \quad C^T = \begin{bmatrix} 1 & -1 \\ -1 & 10^{-4} \end{bmatrix} ;$$

$$A^{-1} = \frac{1}{10^{-4}-1} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 10^{-4} \end{bmatrix} \approx \begin{bmatrix} -1 & 1 \\ 1 & -10^{-4} \end{bmatrix}$$

$$\|A^{-1}\|_\infty \approx 2 ; \quad \|A\|_\infty = 2 ; \kappa(A) \approx 4$$

LU - factorization

$$A = \begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} ; & u_{12} \\ l_{21}u_{11} ; & l_{21} \cdot u_{12} + u_{22} \end{bmatrix}$$

$u_{11} = 10^{-4}$; $u_{12} = 1$; $l_{21} \cdot u_{11} = 1 \rightarrow$

$l_{21} = \frac{1}{u_{11}} = \frac{1}{10^{-4}} = 10^4$; $l_{21} \cdot u_{12} + u_{22} = 1$

$u_{22} = 1 - l_{21} \cdot u_{12} = 1 - 10^4 \cdot 1 \approx -10^4$

$$A = LU = \underbrace{\begin{bmatrix} 1 ; & 0 \\ 10^4 ; & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 10^{-4} ; & 1 \\ 0 ; & -10^4 \end{bmatrix}}_{U} \neq A$$

Ex.2/ Check cond. numbers:

$$L = \begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 10^{-4} & 1 \\ 0 & -10^4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix} \Rightarrow L^{-1} = \frac{1}{\det L} \cdot C^T = C^T$$

$$C = \begin{bmatrix} 1 & -10^4 \\ 0 & 1 \end{bmatrix}; \quad C^T = \begin{bmatrix} 1 & 0 \\ -10^4 & 1 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} 1 & 0 \\ -10^4 & 1 \end{bmatrix}; \quad \|L^{-1}\|_\infty \approx 10^4; \|L\|_\infty \approx 10^4$$

$$U = \begin{bmatrix} 10^{-4} & 1 \\ 0 & -10^4 \end{bmatrix} \Rightarrow U^{-1} = \frac{1}{\det U} C^T = -C^T$$

$$C = \begin{bmatrix} -10^4 & 0 \\ -1 & 10^{-4} \end{bmatrix}; \quad C^T = \begin{bmatrix} -10^4 & -1 \\ 0 & 10^{-4} \end{bmatrix};$$

$$U^{-1} = -C^T = \begin{bmatrix} 10^4 & 1 \\ 0 & -10^{-4} \end{bmatrix}; \quad \|U^{-1}\|_\infty \approx 10^4$$

$$\|U\|_\infty \approx 10^4$$

Observe: $K(L) = \|L^{-1}\|_\infty \cdot \|L\|_\infty = 10^8$!

$K(U) = \|U^{-1}\|_\infty \cdot \|U\|_\infty = 10^8$!

very big cond. numbers!

$\Downarrow$

not ac

3/

Perform pivoting in A

$$A = \begin{bmatrix} 1 & 1 \\ 10^{-4} & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= LU \Rightarrow \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11}; & u_{12} \\ l_{21} \cdot u_{11}; & l_{21} \cdot u_{12} + u_{22} \end{bmatrix}$$

$$u_{11} = a_{11} = 1; \; u_{12} = a_{12} = 1; \; l_{21} \cdot u_{11} = a_{21}$$

$$\Rightarrow l_{21} = 10^{-4}; \; l_{21} \cdot u_{12} + u_{22} = a_{22} \Rightarrow$$

$$u_{22} = a_{22} - l_{21} \cdot u_{12} = 1 - 10^{-4} \cdot 1 \approx 1$$

$$= LU: \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 10^{-4} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{U} \approx A$$

heck cond. numb.

$$L = \begin{bmatrix} 1 & 0 \\ 10^{-4} & 1 \end{bmatrix}; \; L^{-1} = C^T; \; C = \begin{bmatrix} 1 & -10^{-4} \\ 0 & 1 \end{bmatrix}; \; L^{-1} = \begin{bmatrix} 1 & 0 \\ -10^{-4} & 1 \end{bmatrix}$$

$$\| L^{-1} \|_\infty \approx 1; \; \| L \|_\infty \approx 1; \; \kappa(L) \approx 1$$

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \; U^{-1} = C^T; \; C = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \; U^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\| U^{-1} \|_\infty = 2; \; \| U \|_\infty = 2; \; \kappa(U) = 4$$

Error analysis in algorithm of PLU factorization

When $j \leq k$:

$$(1) \quad \boxed{u_{jk} = a_{jk} - \sum_{i=1}^{j-1} l_{ji} u_{ik}}$$

When $j > k$:

$$(2) \quad \boxed{l_{jk} = \frac{a_{jk} - \sum_{i=1}^{k-1} l_{ji} u_{ik}}{u_{kk}}}$$

From (1): compute a_{jk} as with round-off

$$a_{jk} = \frac{1}{1+\delta'} u_{jk} \cdot l_{jj} + \sum_{i=1}^{j-1} l_{ji} u_{ik} (1+\delta_i)$$

$$= (1+\delta_j) u_{jk} \cdot l_{jj} + \sum_{i=1}^{j-1} l_{ji} u_{ik} + \sum_{i=1}^{j-1} l_{ji} u_{ik} \delta_i$$

$$\leq \underbrace{\sum_{i=1}^{j} l_{ji} u_{ik}}_{LU} + \underbrace{\sum_{i=1}^{j} l_{ji} u_{ik} \delta_i}_{\text{Errors}}$$