Machine learning algorithms for inverse problems

Introduction. Physical formulations leading to ill- and well-posed problems

Lecture 1

Organization of the course

- Link for CANVAS course page:

  https://canvas.gu.se/courses/12237000000042894

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- Course coordinator/examinator, registration for PhD students/researchers: Larisa Beilina

  larisa@chalmers.se

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- Course literature: L. Beilina, M. Klibanov, *Approximate global convergence and adaptivity for coefficient inverse problems*. Book, available at

  https://www.springer.com/gp/book/9781441978042

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- The course projects together with examples of Matlab and C++ programs are available for download in CANVAS. Description of the C++/PETSC project *"Solution of time-harmonic acoustic coefficient inverse problem"* together with examples of Matlab and C++ programs is available at the link

  www.waves24.com/download

Types of instruction and assessment

- The material of the course includes online lectures in Zoom (slides and video lectures), description of the computer projects and open source software (Matlab codes, C++/PETSC codes).
- All material of the course is available in CANVAS (contact me for registration):

  `https://canvas.gu.se/courses/122370000000042894`
- Language of instruction is English.
- This course gives 7.5 Hp. The grade Pass (G) or Fail (U) is given in this course.
- The examination consists of the computer project and the final oral exam at the end of the course.

Computer projects

- Examination at this course consists in the oral presentation of the computer project which can be done in groups by 2-4 persons.
- The course projects together with examples of Matlab and C++/PETSc programs are available for download in CANVAS. Description of the C++/PETSC project *"Solution of time-harmonic acoustic coefficient inverse problem"* together with examples of Matlab and C++ programs is available fopr download at the link `www.waves24.com/download`
- You can do also your own project related to the course and the course material. The project should be done using methods studied in the course (for example, using studied regularization strategies or implemented in C++/PETSc).

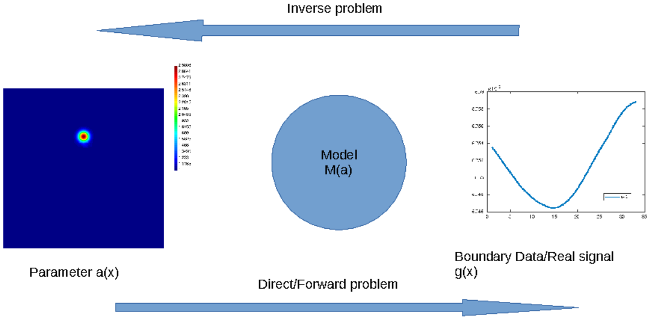
Organization: projects

- To pass this course you should do any computer project presented in CANVAS. The project can be done in groups by 2-4 persons.
- Sent final report for computer project with description of your work together with Matlab or C++/PETSc programs to my e-mail. Report should have description of used techniques, tables and figures confirming your investigations. Analysis of obtained results is necessary to present in section “Numerical examples” and summarize results in the section “Conclusion”.
- The project should be presented for all participants of the course in a short presentation (20 min) at the end of the course. The dates for presentations will be announced in CANVAS. All participants of the course can act as opponents for the project presenters.
- Matlab programs for solution of least squares problem and C++/PETSc programs for solution of Poisson’s equation on a unite square are available for download from the link

  https://github.com/springer-math/Numerical_Linear_Algebra_Theory_and_Applications

- Physical formulations leading to ill- and well-posed problems
- Methods of regularization of inverse and ill-posed problems (Morozov's discrepancy, balancing principle, iterative regularization)
- Numerical methods of solution of inverse and ill-posed problems: methods for image reconstruction and image deblurring, Lagrangian approach and adaptive optimization, methods of analytical reconstruction and layer-stripping algorithms, least-squares algorithms.
- Machine learning classification algorithms and neural networks (perceptron algorithm, least squares, SVM and Kernel methods for classification).

Introduction: Inverse and ill-posed problems

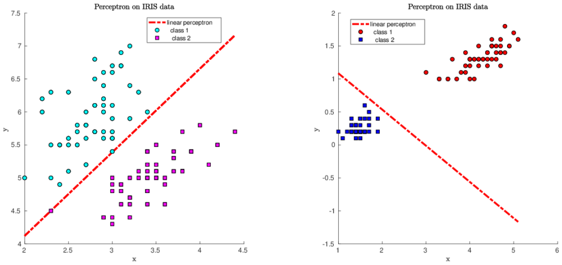


Scheme for solution of CIP for $\triangle u(x) - s^2 a(x) u(x) = -f(x), \partial_n u = 0$.

- Inverse and ill-posed problems include solution of CIPs for PDE, solution of parameter identification problems governed by system of ODE, inverse sourse problems, inverse spectral problems, solution of Fredholm integral equations of the first kind (ill-posed problems).

Classification problems

Classification problems can be considered as the type of inverse problem since the goal of classification is to find optimal vector of weights $\omega = [\omega_1, ..., \omega_n]$ to separate given data x by the decision line $\omega^T x$.



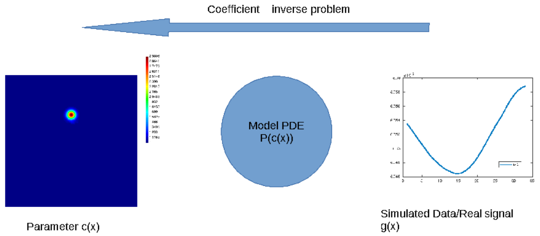
Example of classification: determine the decision line for points presented in the Figure. Two classes are separated by the linear equation with three weights $\omega_i, i = 1, 2, 3$, given by

$$\omega_1 + \omega_2 x + \omega_3 y = 0. \quad (1)$$

Decision lines on the figure computed by the perceptron learning algorithm for separation of two classes using Iris dataset.

Test Matlab program to generate this figures on the course page in CANVAS.

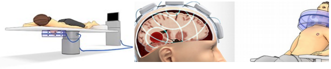
Introduction: Inverse and ill-posed problems



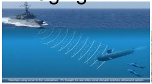
- These applications are modelled by acoustic, elastic or electromagnetic wave eq. which include different physical parameters (wave speed *c* - acoustic equation, elasticity parameters λ and μ - elastic equations, dielectric permittivity ε, magnetic permeability μ, conductivity σ - Maxwell's eq.).
- A coefficient inverse problem for a given PDE aims at estimating a spatially distributed coefficient of the model PDE using measurements taken on the boundary of the domain of interest.

Applications leading to inverse and ill-posed problems

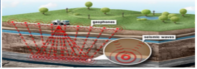
Microwave medical imaging



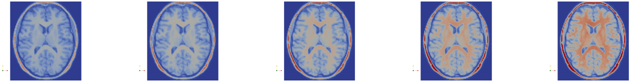
Acoustic imaging



Elastic imaging



Examples of CIPs. Biomedical Imaging at the Department of Electrical Engineering at CTH, Chalmers. Left: breast cancer detection, setup of Stroke Finder; microwave hyperthermia in cancer treatment; Middle: acoustic imaging; right: subsurface imaging.



Example of ill-posed problem: restoration of MRI images for the parietal lobe `http://brain-development.org/`

- Inverse and ill-posed problems arise in many real-world applications including medical microwave, optical and ultrasound imaging, MRT, MRI, oil prospecting and shape reconstruction, nondestructive testing of materials and detection of explosives, seeing through the walls and constructing of new materials.
- Physical applications are modelled by acoustic, elastic or electromagnetic wave eq. which include different physical parameters s. t. wave speed c - acoustic equation; elasticity parameters λ and μ - elastic equations; dielectric permittivity ε, magnetic permeability μ, conductivity σ - Maxwell's eq.
- A coefficient inverse problem for a given model PDE aims at estimating a spatially distributed coefficient of the model PDE using measurements taken on the boundary of the domain of interest.

Notations and Definitions

- The theory of Ill-Posed Problems addresses the following fundamental question: *How to obtain a good approximation for the solution of an ill-posed problem in a stable way?*
- A numerical method, which provides a stable and accurate solution of an ill-posed problem, is called the *regularization* method for this problem.
- Foundations of the theory of Ill-Posed Problems were established by three Russian mathematicians: A. N. Tikhonov [T1,TA,T], M.M. Lavrent'ev [L] and V. K. Ivanov [I] in 1960-ies. The first foundational work was published by Tikhonov in 1943 [T].

[T1] A. N. Tikhonov, On the stability of inverse problems, *Doklady of the USSR Academy of Science*, 39, 195-198, 1943

[TA] A. N. Tikhonov and V. Y. Arsenin. *Solutions of Ill-Posed Problems,* Winston and Sons, Washington, DC, 1977.

[T] A.N. Tikhonov, A.V. Goncharsky, V.V. Stepanov and A.G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, London: Kluwer, London, 1995.

[L] M.M. Lavrentiev, *Some Improperly Posed Problems of Mathematical Physics*, Springer, New York, 1967.

[I] V. K. Ivanov, On ill-posed problems, *Mat. USSR Sb.*, 61, 211–223, 1963.

- Theory of inverse and ill-posed problems is developed further and a lot of new works on this subject are available, some of them are:

- S. Arridge, Optical tomography in medical imaging, *Inverse Problems*, 15, 841–893, 1999.
- A.B. Bakushinsky and M.Yu. Kokurin, *Iterative Methods for Approximate Solution of Inverse Problems*, Springer, New York, 2004.
- F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, New York, 2006.
- K. Chadan and P. Sabatier, Inverse Problems in Quantum Scattering Theory, Springer, New York, 1989.
- G. Chavent, *Nonlinear Least Squares for Inverse Problems: Theoretical Foundations and Step-by-Step Guide for Applications (Scientific Computation)*, Springer, New York, 2009.
- V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, New York, 2005.
- B. Kaltenbacher, A. Neubauer and O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*, de Gruyter, New York, 2008.
- A. Kirsch, *An Introduction To the Mathematical Theory of Inverse Problems*, Springer, New York, 2011.
- K. Ito, B. Jin, Inverse Problems: Tikhonov theory and algorithms, Series on Applied Mathematics, V.22, World Scientific, 2015.
- Tikhonov, A.N., Goncharsky, A., Stepanov, V.V., Yagola, A.G., Numerical Methods for the Solution of Ill-Posed Problems,ISBN 978-94-015-8480-7, 1995.

Notations and Definitions

Let $u(x)\,,\; x = (x_1, ..., x_n) \in \Omega$ be a k times continuously differentiable function defined in Ω. Denote

$$D^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n}, \;\; |\alpha| = \alpha_1 + \ldots + \alpha_n$$

the partial derivative of the order $|\alpha| \leq k$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with integers $\alpha_i \geq 0$. Denote $C^k\left(\overline{\Omega}\right)$ the Banach space of functions $u(x)$ which are continuous in the closure $\overline{\Omega}$ of the domain Ω together with their derivatives $D^{\alpha} u, |\alpha| \leq m$. The norm in this space is defined as

$$\|u\|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} \left| D^{\alpha} u(x) \right| < \infty.$$

Notations and Definitions

By definition $C^0\left(\overline{\Omega}\right) = C\left(\overline{\Omega}\right)$ is the space of functions continuous in $\overline{\Omega}$ with the norm

$$\|u\|_{C\left(\overline{\Omega}\right)} = \sup_{x \in \Omega} \left| u\left(x\right) \right|.$$

We also introduce Hölder spaces $C^{k+\alpha}\left(\overline{\Omega}\right)$ for any number $\alpha \in \left(0, 1\right)$. The norm in this space is defined as

$$\|u\|_{C^{k+\alpha}\left(\overline{\Omega}\right)} := |u|_{k+\alpha} := \|u\|_{C^k\left(\overline{\Omega}\right)} + \sup_{x,y \in \Omega,\, x \neq y} \frac{\left| u\left(x\right) - u\left(y\right) \right|}{|x - y|^{\alpha}},$$

provided that the last term is finite. It is clear that if the function $u \in C^{k+1}\left(\overline{\Omega}\right)$, then $u \in C^{k+\alpha}\left(\overline{\Omega}\right), \forall \alpha \in \left(0, 1\right)$ and

$$|u|_{k+\alpha} \leq C \|u\|_{C^{k+1}\left(\overline{\Omega}\right)}, \ \forall u \in C^{k+1}\left(\overline{\Omega}\right),$$

where $C = C\left(\Omega, \alpha\right) > 0$ is a constant independent on the function u.

Notations and Definitions

Consider the Sobolev space $H^k(\Omega)$ of all functions with the norm defined as

$$\|u\|^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^2 \, dx < \infty,$$

where $D^\alpha u$ are weak derivatives of the function u. By the definition $H^0(\Omega) = L_2(\Omega)$. It is well known that $H^k(\Omega)$ is a Hilbert space with the inner product defined as

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha u D^\alpha v dx.$$

Notations and Definitions

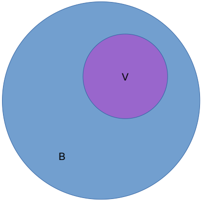
Let $T > 0$ and $\Gamma \subseteq \partial\Omega$ be a part of the boundary $\partial\Omega$ of the domain Ω. We will use the following notations

$$Q_T = \Omega \times (0, T)\,, S_T = \partial\Omega \times (0, T)\,, \Gamma_T = \Gamma \times (0, T)\,, D_T^{n+1} = \mathbb{R}^n \times (0, T)\,.$$

The space $C^{2k,k}\left(\overline{Q}_T\right)$ is defined as the set of all functions $u\,(x, t)$ having derivatives $D_x^\alpha D_t^\beta u \in C\left(\overline{Q}_T\right)$ with $|\alpha| + 2\beta \leq 2k$ and with the following norm

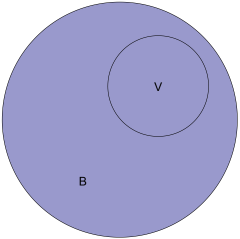
$$\|u\|_{C^{2k,k}\left(\overline{Q}_T\right)} = \sum_{|\alpha|+2\beta\leq 2k} \max_{\overline{Q}_T} \left|D_x^\alpha D_t^\beta u\,(x, t)\right|.$$

Notations and Definitions



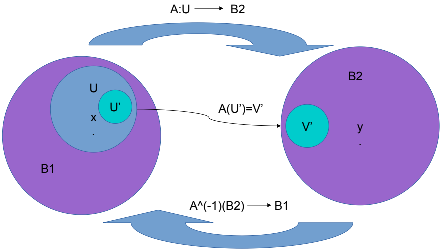
Definition 1. Let B be a Banach space. The set $V \subset B$ is called *precompact* set if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence (i.e., the Cauchy subsequence).

Although by the Cauchy criterion the subsequence in this Definition 1 converges to a certain point, there is no guarantee that this point belongs to the set V. If we consider the closure of V, i.e. the set $\overline{V}$, then all limiting points of all convergent sequences in V would belong to $\overline{V}$.



Definition 2. Let B be a Banach space. The set $V \subset B$ is called *compact set* if V is a closed set, $V = \overline{V}$, every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence and the limiting point of this subsequence belongs to the set V.

Notations and Definitions



Definition 3. Let B_1 and B_2 be two Banach spaces, $U \subseteq B_1$ be a set and $A : U \to B_2$ be a continuous operator. The operator A is called a *compact operator* or *completely continuous* operator if it maps any bounded subset $U' \subseteq U$ in a precompact set in B_2. Clearly if U' is a closed set, then $A(U')$ is a compact set.

Notations and Definitions. Ascoli-Archela theorem

The following theorem is well known under the name of Ascoli-Archela theorem (More general formulations of this theorem can also be found).
Theorem *The set of functions $\mathcal{M} \subset C\left(\overline{\Omega}\right)$ is a compact set if and only if it is uniformly bounded and equicontinuous. In other words, if the following two conditions are satisfied:*
1. There exists a constant $M > 0$ such that

$$\|f\|_{C\left(\overline{\Omega}\right)} \leq M, \ \forall f \in \mathcal{M}.$$

2. For any $\varepsilon > 0$ there exists $\delta = \delta\left(\varepsilon\right) > 0$ such that

$$\left|f\left(x\right) - f\left(y\right)\right| < \varepsilon, \quad \forall x, y \in \{|x - y| < \delta\} \cap \overline{\Omega}, \ \forall f \in \mathcal{M}.$$

Classical Correctness and Conditional Correctness

The notion of the classical correctness is called sometimes *Correctness by Hadamard*.

Definition. Let B_1 and B_2 be two Banach spaces. Let $G \subseteq B_1$ be an open set and $F : G \to B_2$ be an operator. Consider the equation

$$F(x) = y, \ x \in G. \tag{2}$$

The problem of solution of equation (2) is called *well-posed by Hadamard*, or simply *well-posed*, or *classically well-posed* if the following three conditions are satisfied:

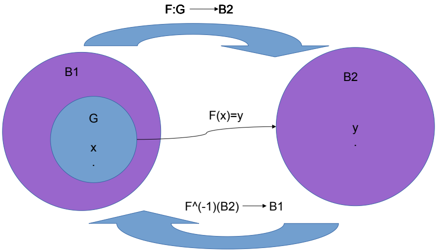
1. For any $y \in B_2$ there exists a solution $x = x(y)$ of equation (2) (existence theorem).

2. This solution is unique (uniqueness theorem).

3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

If equation (2) does not satisfy to at least one these three conditions, then the problem (2) is called *ill-posed*.

Classical Correctness



The problem is *classically well-posed* if:

1. For any $y \in B_2$ there exists a solution $x = x(y)$ of $F(x) = y$.

2. This solution is unique (uniqueness theorem).

3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

We say that the right hand side of equation

$$F(x) = y,\ x \in G. \tag{3}$$

is given with an error of the level $\delta > 0$ (small) if $\|y^* - y\|_{B_2} \leq \delta$, where y^* is the exact value.

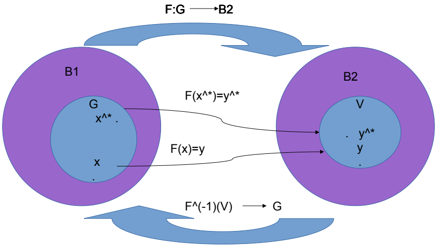
Definition Let B_1 and B_2 be two Banach spaces. Let $G \subset B_1$ be an *a priori* chosen set of the form $G = \overline{G_1}$, where G_1 is an open set in B_1. Let $F : G \to B_2$ be a continuos operator. Suppose that $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here y^* is the ideal noiseless data, y_δ is noisy data . The problem (3) is called *conditionally well-posed on the set* G, or *well-posed by Tikhonov* on the set G if the following three conditions are satisfied:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

Conditional Correctness



The problem (3) is called *conditionally well-posed on the set* G if:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

The Fundamental Concept of Tikhonov

This concept consists of the following three conditions which should be in place when solving the ill-posed problem (3):

1. One should *a priori* assume that there exists an ideal exact solution x^* of equation (3) for an ideal noiseless data y^*.

2. The correctness set G should be chosen *a priori*, meaning that some *a priori* bounds imposed on the solution x of equation (3) should be imposed.

3. To construct a stable numerical method for the problem (3), one should assume that there exists a family $\{y_\delta\}$ of right hand sides of equation (3), where $\delta > 0$ is the level of the error in the data with $||y^* - y_\delta||_{B_2} \leq \delta$. Next, one should construct a family of approximate solutions $\{x_\delta\}$ of equation (3), where x_δ corresponds to y_δ. The family $\{x_\delta\}$ should be such that

$$\lim_{\delta \to 0^+} ||x_\delta - x^*|| = 0.$$

Quasi-solution

Another approach to the solution of ill-posed problem is concept pf quasi-solution. This concept was introduced by Ivanov in 1962 in the work [Ivanov, 1962]. Let A be a compact operator, $x \in M$, M is a compact set such that $M \subset Q$, $f \in A(M) \subset F$: Then approximate solution of the problem

$$Ax = f$$

can be obtained by

$$x = A^{-1} f.$$

for small perturbations in the rhs f.
The main point here is that $f \in A(M) \subset F$ otherwise the solution can not be obtained by $x = A^{-1} f$. Since it is difficult to check if $f \in A(M) \subset F$ then it was introduced the concept of quasi-solution.

V.K.Ivanov, On linear problems which are not well-posed, Dokl.Akad.Nauk SSSR, 145(2), 211-223. 1962 (In Russian)

Quasi-solution

Definition (Ivanov, 1962)
A quasi-solution to the equation

$$Ax = f \tag{4}$$

on a set $M \subset Q$ is an element $x_K \in M$ that minimizes the residual

$$R(Ax_k, f) = \inf_{x \in M} R(Ax, f) \tag{5}$$

If M is a compact set then there exists a quasi-solution for any $f \in F$.
If in addition $f \in A(M)$ the quasi-solutions x_k (it can be a lot of such solutions) are the same as exact solution x.
Here is a sufficient condition for a quasi-solution to be unique and continuously depend on the rhs f.

Quasi-solution

- **Theorem** [Ivanov, 2002, Tikhonov Arsenin, 1974]
  Assume that the equation (4) has at most one solution on a compact set M and $\forall f \in M$ the projection Pf into $A(M)$ is unique. Then a quasi-solution of the equation (4) is unique and continuously depends on f.
- We can conclude that the problem of finding a quasi-solution on a compact set is well-posed problem.
- If the quasi-solution is not unique, then its quasi-solutions form a subset of the compact set M and in this case this set continuously depends on f (Ivanov, 1963).

V. K. Ivanov, V. V. Vasin, V. P. Tanana, *Theory of linear ill-posed problems and its applications*, VSP, Utrecht, 2002.
A. N. Tikhonov, V. Ya. Arsenin, *Solutions of ill-posed problems*, Wiley, 1977.

Ill-posed problem: differentiation of a function given with a noise

Suppose that the function $f(x), x \in [0,1]$ is given with a noise, i.e. suppose that instead of $f(x) \in C^1[0,1]$ the following function $f_\delta(x)$ is given

$$f_\delta(x) = f(x) + \delta f(x), x \in [0,1],$$

where $\delta f(x)$ is the noisy component. Let $\delta > 0$ be a small parameter such that $||\delta f||_{C[0,1]} \leq \delta$. Let us show that the problem of calculating the derivative $f_\delta'(x)$ is unstable.

Examples of ill-posed problems. Differentiation of a function given with a noise

For example, take

$$\delta f(x) = \frac{\sin(n^2 x)}{n},$$

where $n > 0$ is a large integer. Then the $C[0,1]$-norm of the noisy component is small,

$$||\delta f||_{C[0,1]} \leq \frac{1}{n}.$$

However, the difference between derivatives of noisy and exact functions

$$f'_\delta(x) - f'(x) = \delta f'(x) = n \cos n^2 x$$

is not small in any reasonable norm.

Ill-posed problem: differentiation of a function given with a noise

A simple regularization method of stable calculation of derivatives is that the step size h in the corresponding finite difference discretization should be connected with the level of noise δ.

$$f'_\delta(x) \approx \frac{f(x+h) - f(x)}{h} + \frac{\delta f(x+h) - \delta f(x)}{h}. \tag{6}$$

The first term in the right hand side of (6) is close to the exact derivative $f'(x)$, if h is small enough. The second term, however, comes from the noise and we need to balance these two terms via an appropriate choice of $h = h(\delta)$.

$$\left| f'_\delta(x) - \frac{f(x+h) - f(x)}{h} \right| = \left| \frac{\delta f(x+h) - \delta f(x)}{h} \right| \le \frac{2\delta}{h}.$$

Hence, we should choose $h = h(\delta)$ such that

$$\lim_{\delta \to 0} \frac{2\delta}{h(\delta)} = 0.$$

Ill-posed problem: differentiation of a function given with a noise

For example, let $h(\delta) = \delta^{\mu}$, where $\mu \in (0,1)$. Then

$$\lim_{\delta \to 0} \left| f'_{\delta}(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h} = \frac{2\delta}{\delta^{\mu}} \leq \lim_{\delta \to 0} \left(2\delta^{1-\mu} \right) = 0.$$

Hence, the problem becomes stable for this choice of the grid step size $h(\delta) = \delta^{\mu}$. This means that $h(\delta)$ is the regularization parameter.

Ill-posed problem: integral equation of the first kind

- Let $\Omega \subset \mathbb{R}^n$ is a bounded domain and the function $K(x, y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Recall that the equation

$$g(x) + \int\limits_{\Omega} K(x, y) f(y) dy = f(x), x \in \Omega, \tag{7}$$

  is called *integral equation of the second kind*. The problem is to find function $f(x)$ by known $K(x, y)$ and $g(x)$. These equations are considered quite often in the classic theory of PDEs and are solved by Liouville-Neumann (iterative) series.
- Next, let $\Omega' \subset \mathbb{R}^n$ be a bounded domain and the function $K(x, y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Unlike (7), the equation

$$\int\limits_{\Omega} K(x, y) f(y) dy = p(x), x \in \Omega' \tag{8}$$

  is called the integral equation of the first kind. The Fredholm theory does not work for such equations. The problem of solution of equation (8) is an ill-posed problem.

Ill-posed problem: integral equation of the first kind

Consider equation (8):

$$\int_{\Omega} K(x, y) f(y) \, dy = p(x), x \in \Omega'$$

The function $K(x, y)$ is called *kernel* of the integral operator. Equation (8) can be rewritten in the form

$$Af = p, \tag{9}$$

where $A : C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}'\right)$ is the integral operator in (8). It is well known from the standard Functional Analysis course that A is a compact operator. We now show that the problem (9) is an ill-posed problem.

Example of an integral equation of the first kind

Let $\Omega = (0,1), \Omega' = (a,b)$. Let $f_n(x) = f(x) + \sin nx$. Then for $x \in (0,1)$

$$\int_0^1 K(x,y) f_n(y)\, dy = \int_0^1 K(x,y) f(y)\, dy + \int_0^1 K(x,y) \sin ny dy = g_n(x), \tag{10}$$

where $g_n(x) = p(x) + p_n(x)$ and

$$p_n(x) = \int_0^1 K(x,y) \sin ny dy.$$

By the Lebesque lemma

$$\lim_{n\to\infty} \|p_n\|_{C[a,b]} = 0.$$

However, it is clear that

$$\left\| f_n(x) - f(x) \right\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$$

is not small for large n.

Ill-posed problem: the case of a general compact operator

Let H_1 and H_2 be two Hilbert spaces with $\dim H_1 = \dim H_2 = \infty$. Remind that a sphere in an infinitely dimensional Hilbert space is not a compact set.

Theorem (Theorem 1.2 of [BK]) *Let* $G = \left\{ \|x\|_{H_1} \leq 1 \right\} \subset H_1$, G is not a compact set. *Let* $A : G \to H_2$ *be a compact operator and let* $R(A) := A(G)$ *be its range. Consider an arbitrary point* $y_0 \in R(A)$. *Let* $\varepsilon > 0$ *be a number and* $U_\varepsilon(y_0) = \left\{ y \in H_2 : \|y - y_0\|_{H_2} < \varepsilon \right\}$. *Then there exists a point* $y \in U_\varepsilon(y_0) \diagdown R(A)$. *If, in addition, the operator* A *is one-to-one, then the inverse operator* $A^{-1} : R(A) \to G$ *is not continuous. Hence, the problem of the solution of the equation*

$$A(x) = z, x \in G, z \in R(A) \tag{11}$$

is unstable, i.e. this is an ill-posed problem.

Tikhonov's theorem

Theorem (Tikhonov, 1943). *Let B_1 and B_2 be two Banach spaces. Let $U \subset B_1$ be a compact set and $F : U \to B_2$ be a continuous operator. Assume that the operator F is one-to-one. Let $V = F(U)$. Then the inverse operator $F^{-1} : V \to U$ is continuous.*

Proof. Assume the opposite: that the operator F^{-1} is not continuous on the set V. Then there exists a point $y_0 \in V$ and a number $\varepsilon > 0$ such that for any $\delta > 0$ there exists a point y_δ such that although $\|y_\delta - y_0\|_{B_2} < \delta$, still $\left\|F^{-1}(y_\delta) - F^{-1}(y_0)\right\|_{B_1} \geq \varepsilon$. Hence, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \delta_n = 0^+$ and the corresponding sequence $\{y_n\}_{n=1}^{\infty} \subset V$ such that

$$\left\|y_{\delta_n} - y_0\right\|_{B_2} < \delta_n, \| \underbrace{F^{-1}(y_n)}_{x_n} - \underbrace{F^{-1}(y_0)}_{x_0} \|_{B_1} \geq \varepsilon. \tag{12}$$

Denote

$$x_n = F^{-1}(y_n), x_0 = F^{-1}(y_0). \quad (13)$$

Then by (12) we have

$$\|x_n - x_0\|_{B_1} \geq \varepsilon. \quad (14)$$

Since U is a compact set and all points $x_n \in U$, then one can extract a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ from the sequence $\{x_n\}_{n=1}^{\infty}$. Let $\lim_{k\to\infty} x_{n_k} = \overline{x}$. Then $\overline{x} \in U$. Since $F(x_{n_k}) = y_{n_k}$ and the operator F is continuous, then by (12) and (13) we have:

$$\begin{aligned} x_n &= F^{-1}(y_n) \Rightarrow F(x_n) = y_n, \\ x_0 &= F^{-1}(y_0) \Rightarrow F(x_0) = y_0; \\ F(\bar{x}) &= \lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = y_0 \end{aligned} \quad (15)$$

So, we obtained, that $F(x_0) = y_0$ and $F(\overline{x}) = y_0$. Since the operator F is one-to one, we should have $\overline{x} = x_0$. However, by (14) $\left\|\overline{x} - x_0\right\|_{B_1} \geq \varepsilon$. We got a contradiction. □

Model inverse problems

We will consider now following model inverse problems:

- Elliptic inverse problems
  - Elliptic CIPs
  - Cauchy problem
  - Inverse source problem
  - Inverse spectral problem
- Hyperbolic CIPs
- Parabolic CIPs
- Determination of the initial condition in hyperbolic or parabolic PDE

Elliptic inverse problems

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ be a domain with a boundary Γ.
We will present several inverse problems for a second order elliptic PDE

$$-\nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u + p(x)u = f(x), \quad x \in \Omega \tag{16}$$

We can consider the equation (16) with suitable b.c. (for example, Dirichlet or Neumann b.c.).
Functions $a(x), b(x)$ and $p(x)$ are known as

- $a(x)$ - conductivity or diffusion coefficient
- $b(x)$ - convection coefficient
- $p(x)$ potential coefficient
- $f(x)$ - the source term

$$-\nabla\cdot(a(x)\nabla u)+b(x)\cdot\nabla u+p(x)u=f(x),\quad x\in\Omega \tag{17}$$

Let the function $u(x)\in C^2$ satisfies to the (17) and

$$u|_\Gamma=p\left(x\right),\quad \frac{\partial u}{\partial n}|_\Gamma=q\left(x\right). \tag{18}$$

The Elliptic Coefficient Inverse Problem. Suppose that one of coefficients in equation (17) is unknown inside of the domain Ω and is known outside of it. Assume that all other coefficients in (17) are known. Determine that unknown coefficient inside of Ω, assuming that the functions $p\left(x\right)$ and $q\left(x\right)$ in (18) are known.

Cauchy problem

Cauchy problem arises, for example, in electrocardiography and geophysical prospectation. This problem is severally ill-posed and lacks a continuous dependence on data.
Let Γ_c and $\Gamma_i = \Gamma \setminus \Gamma_c$ be two disjoint parts of the boundary Γ. Here,

- Γ_c - observation boundary
- Γ_i - boundary, where observations are not taken

The Cauchy problem reads: given the Cauchy data g and h on the boundary Γ_c, find the function u on the boundary Γ_i, or:

$$-\nabla \cdot (a(x)\nabla u) = 0, x \in \Omega, \tag{19}$$

$$u = g, \quad x \in \Gamma_c, \tag{20}$$

$$a\frac{\partial u}{\partial n} = h, \quad x \in \Gamma_c. \tag{21}$$

Hadamard's example for the Cauchy problem

This example shows that the Cauchy problem for Laplace equation does't depends continuously on data. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and the boundary $\Gamma_c = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Consider the solution $u = u_n, n = 1, 2, ...$ to the Cauchy problem

$$\triangle u = 0, \quad x \in \Omega, \tag{22}$$

$$u = 0, \quad x \in \Gamma_c, \tag{23}$$

$$\frac{\partial u}{\partial n} = -n^{-1} \sin nx_1, \quad x \in \Gamma_c. \tag{24}$$

The function

$$u_n = n^{-2} \sin nx_1 \sinh nx_2 = n^{-2} \sin nx_1 (e^{nx_2} - e^{-nx_2})/2$$

is the solution of the problem (22) and it is a unique solution (by Holmgren's theorem for Laplace equation).
We observe that on Γ_c we have $\lim_{n\to\infty} \frac{\partial u_n}{\partial n} = 0$. However, for all $x_2 > 0$ the solution $\lim_{n\to\infty} u_n(x_1, x_2) = \lim_{n\to\infty} n^{-2} \sin nx_1 \sinh nx_2 = \lim_{n\to\infty} n^{-2} \sin nx_1 (e^{nx_2} - e^{-nx_2})/2 = \infty$.