JSS30, Summer School, COM5: Machine learning in inverse and ill-posed problems

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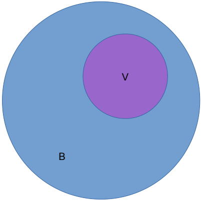
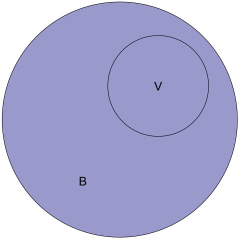
Physical formulations leading to ill- and well-posed problems
Lecture 2

Notations and Definitions



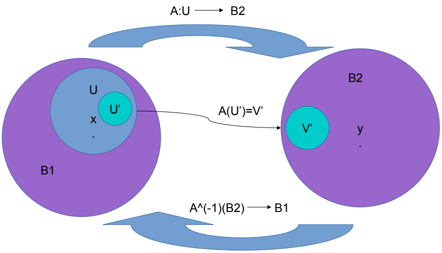
Definition 1. Let B be a Banach space. The set $V \subset B$ is called *precompact* set if every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence (i.e., the Cauchy subsequence).
Although by the Cauchy criterion the subsequence in this Definition 1 converges to a certain point, there is no guarantee that this point belongs to the set V. If we consider the closure of V, i.e. the set $\overline{V}$, then all limiting points of all convergent sequences in V would belong to $\overline{V}$.

Notations and Definitions



Definition 2. Let B be a Banach space. The set $V \subset B$ is called *compact* set if V is a closed set, $V = \overline{V}$, every sequence $\{x_n\}_{n=1}^{\infty} \subseteq V$ contains a fundamental subsequence and the limiting point of this subsequence belongs to the set V.

Notations and Definitions



Definition 3. Let B_1 and B_2 be two Banach spaces, $U \subseteq B_1$ be a set and $A : U \rightarrow B_2$ be a continuous operator. The operator A is called a *compact operator* or *completely continuous* operator if it maps any bounded subset $U' \subseteq U$ in a precompact set in B_2. Clearly if U' is a closed set, then $A(U')$ is a compact set.

Notations and Definitions. Ascoli-Archela theorem

The following theorem is well known under the name of Ascoli-Archela theorem (More general formulations of this theorem can also be found).

Theorem *The set of functions $\mathcal{M} \subset C\left(\overline{\Omega}\right)$ is a compact set if and only if it is uniformly bounded and equicontinuous. In other words, if the following two conditions are satisfied:*

1. There exists a constant $M > 0$ such that

$$\|f\|_{C(\overline{\Omega})} \leq M, \ \forall f \in \mathcal{M}.$$

2. For any $\varepsilon > 0$ there exists $\delta = \delta\left(\varepsilon\right) > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \forall x, y \in \{|x - y| < \delta\} \cap \overline{\Omega}, \ \forall f \in \mathcal{M}.$$

Classical Correctness and Conditional Correctness

The notion of the classical correctness is called sometimes *Correctness by Hadamard*.

Definition. Let B_1 and B_2 be two Banach spaces. Let $G \subseteq B_1$ be an open set and $F : G \to B_2$ be an operator. Consider the equation

$$F(x) = y, \; x \in G. \tag{1}$$

The problem of solution of equation (1) is called *well-posed by Hadamard*, or simply *well-posed*, or *classically well-posed* if the following three conditions are satisfied:

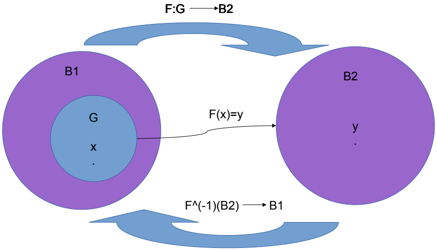
1. For any $y \in B_2$ there exists a solution $x = x(y)$ of equation (1) (existence theorem).

2. This solution is unique (uniqueness theorem).

3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

If equation (1) does not satisfy to at least one these three conditions, then the problem (1) is called *ill-posed*.

Classical Correctness



The problem is *classically well-posed* if:

1. For any $y \in B_2$ there exists a solution $x = x(y)$ of $F(x) = y$.

2. This solution is unique (uniqueness theorem).

3. The solution $x(y)$ depends continuously on y. In other words, the operator $F^{-1} : B_2 \to B_1$ is continuous.

Classical Correctness and Conditional Correctness

We say that the right hand side of equation

$$F(x) = y,\ x \in G. \tag{2}$$

is given with an error of the level $\delta > 0$ (small) if $\|y^* - y\|_{B_2} \leq \delta$, where y^* is the exact value.

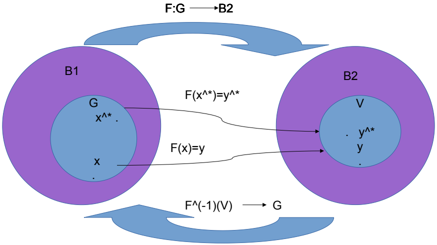
Definition Let B_1 and B_2 be two Banach spaces. Let $G \subset B_1$ be an *a priori* chosen set of the form $G = \overline{G}_1$, where G_1 is an open set in B_1. Let $F : G \to B_2$ be a continuos operator. Suppose that $\|y^* - y_\delta\|_{B_2} \leq \delta$. Here y^* is the ideal noiseless data, y_δ is noisy data . The problem (2) is called *conditionally well-posed on the set* G, or *well-posed by Tikhonov* on the set G if the following three conditions are satisfied:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

Conditional Correctness



The problem (2) is called *conditionally well-posed on the set* G if:

1. It is *a priori* known that there exists an ideal solution $x^* = x^*(y^*) \in G$ of this problem for the ideal noiseless data y^*.

2. The operator $F : G \to B_2$ is one-to-one.

3. The inverse operator F^{-1} is continuous on the set $F(G)$.

The Fundamental Concept of Tikhonov

This concept consists of the following three conditions which should be in place when solving the ill-posed problem (2):

1. One should *a priori* assume that there exists an ideal exact solution x^* of equation (2) for an ideal noiseless data y^*.

2. The correctness set G should be chosen *a priori*, meaning that some *a priori* bounds imposed on the solution x of equation (2) should be imposed.

3. To construct a stable numerical method for the problem (2), one should assume that there exists a family $\{y_\delta\}$ of right hand sides of equation (2), where $\delta > 0$ is the level of the error in the data with $\|y^* - y_\delta\|_{B_2} \leq \delta$. Next, one should construct a family of approximate solutions $\{x_\delta\}$ of equation (2), where x_δ corresponds to y_δ. The family $\{x_\delta\}$ should be such that

$$\lim_{\delta \to 0^+} \|x_\delta - x^*\| = 0.$$

Quasi-solution

Another approach to the solution of ill-posed problem is concept pf quasi-solution. This concept was introduced by Ivanov in 1962 in the work [Ivanov, 1962]. Let A be a compact operator, $x \in M$, M is a compact set such that $M \subset Q$, $f \in A(M) \subset F$: Then approximate solution of the problem

$$Ax = f$$

can be obtained by

$$x = A^{-1}f.$$

for small perturbations in the rhs f.
The main point here is that $f \in A(M) \subset F$ otherwise the solution can not be obtained by $x = A^{-1}f$. Since it is difficult to check if $f \in A(M) \subset F$ then it was introduced the concept of quasi-solution.

V.K.Ivanov, On linear problems which are not well-posed, Dokl.Akad.Nauk SSSR, 145(2), 211-223. 1962 (In Russian)

Quasi-solution

Definition (Ivanov, 1962)
A quasi-solution to the equation

$$Ax = f \tag{3}$$

on a set $M \subset Q$ is an element $x_K \in M$ that minimizes the residual

$$R(Ax_k, f) = \inf_{x \in M} R(Ax, f) \tag{4}$$

If M is a compact set then there exists a quasi-solution for any $f \in F$.
If in addition $f \in A(M)$ the quasi-solutions x_k (it can be a lot of such solutions) are the same as exact solution x.
Here is a sufficient condition for a quasi-solution to be unique and continuously depend on the rhs f.

Quasi-solution

- **Theorem** [Ivanov, 2002, Tikhonov Arsenin, 1974]

  Assume that the equation (3) has at most one solution on a compact set M and $\forall f \in M$ the projection Pf into $A(M)$ is unique. Then a quasi-solution of the equation (3) is unique and continuously depends on f.

- We can conclude that the problem of finding a quasi-solution on a compact set is well-posed problem.

- If the quasi-solution is not unique, then its quasi-solutions form a subset of the compact set M and in this case this set continuously depends on f (Ivanov, 1963).

V. K. Ivanov, V. V. Vasin, V. P. Tanana, *Theory of linear ill-posed problems and its applications*, VSP, Utrecht, 2002.
A. N. Tikhonov, V. Ya. Arsenin, *Solutions of ill-posed problems*, Wiley, 1977.

Ill-posed problem: differentiation of a function given with a noise

Suppose that the function $f(x), x \in [0,1]$ is given with a noise, i.e. suppose that instead of $f(x) \in C^1[0,1]$ the following function $f_\delta(x)$ is given

$$f_\delta(x) = f(x) + \delta f(x), x \in [0,1],$$

where $\delta f(x)$ is the noisy component. Let $\delta > 0$ be a small parameter such that $\|\delta f\|_{C[0,1]} \leq \delta$. Let us show that the problem of calculating the derivative $f_\delta'(x)$ is unstable.

Examples of ill-posed problems. Differentiation of a function given with a noise

For example, take

$$\delta f(x) = \frac{\sin(n^2 x)}{n},$$

where $n > 0$ is a large integer. Then the $C[0,1]$-norm of the noisy component is small,

$$\|\delta f\|_{C[0,1]} \leq \frac{1}{n}.$$

However, the difference between derivatives of noisy and exact functions

$$f'_{\delta}(x) - f'(x) = \delta f'(x) = n \cos n^2 x$$

is not small in any reasonable norm.

Ill-posed problem: differentiation of a function given with a noise

A simple regularization method of stable calculation of derivatives is that the step size h in the corresponding finite difference discretization should be connected with the level of noise δ.

$$f_{\delta}'(x) \approx \frac{f(x+h) - f(x)}{h} + \frac{\delta f(x+h) - \delta f(x)}{h}. \tag{5}$$

The first term in the right hand side of (5) is close to the exact derivative $f'(x)$, if h is small enough. The second term, however, comes from the noise and we need to balance these two terms via an appropriate choice of $h = h(\delta)$.

$$\left| f_{\delta}'(x) - \frac{f(x+h) - f(x)}{h} \right| = \left| \frac{\delta f(x+h) - \delta f(x)}{h} \right| \leq \frac{2\delta}{h}.$$

Hence, we should choose $h = h(\delta)$ such that

$$\lim_{\delta \to 0} \frac{2\delta}{h(\delta)} = 0.$$

Ill-posed problem: differentiation of a function given with a noise

For example, let $h(\delta) = \delta^{\mu}$, where $\mu \in (0,1)$. Then

$$\lim_{\delta \to 0} \left| f'_{\delta}(x) - \frac{f(x+h) - f(x)}{h} \right| \leq \frac{2\delta}{h} = \frac{2\delta}{\delta^{\mu}} \leq \lim_{\delta \to 0} \left(2\delta^{1-\mu} \right) = 0.$$

Hence, the problem becomes stable for this choice of the grid step size $h(\delta) = \delta^{\mu}$. This means that $h(\delta)$ is the regularization parameter.

Ill-posed problem: integral equation of the first kind

- Let $\Omega \subset \mathbb{R}^n$ is a bounded domain and the function $K(x, y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Recall that the equation

$$g(x) + \int\limits_{\Omega} K(x, y) f(y) dy = f(x), x \in \Omega, \tag{6}$$

is called *integral equation of the second kind*. The problem is to find function $f(x)$ by known $K(x, y)$ and $g(x)$. These equations are considered quite often in the classic theory of PDEs and are solved by Liouville-Neumann (iterative) series.

- Next, let $\Omega' \subset \mathbb{R}^n$ be a bounded domain and the function $K(x, y) \in C\left(\overline{\Omega} \times \overline{\Omega}\right)$. Unlike (6), the equation

$$\int\limits_{\Omega} K(x, y) f(y) dy = p(x), x \in \Omega' \tag{7}$$

is called the integral equation of the first kind. The Fredholm theory does not work for such equations. The problem of solution of equation (7) is an ill-posed problem.

Ill-posed problem: integral equation of the first kind

Consider equation (7):

$$\int_{\Omega} K(x,y)\, f(y)\, dy = p(x)\,, x \in \Omega'$$

The function $K(x,y)$ is called *kernel* of the integral operator. Equation (7) can be rewritten in the form

$$Af = p, \tag{8}$$

where $A : C(\overline{\Omega}) \to C\left(\overline{\Omega}'\right)$ is the integral operator in (7). It is well known from the standard Functional Analysis course that A is a compact operator. We now show that the problem (8) is an ill-posed problem.

Example of an integral equation of the first kind

Let $\Omega = (0,1)\,, \Omega' = (a,b)\,.$ Let $f_n(x) = f(x) + \sin nx$. Then for $x \in (0,1)$

$$\int_0^1 K(x,y)\, f_n(y)\, dy = \int_0^1 K(x,y)\, f(y)\, dy + \int_0^1 K(x,y) \sin ny dy = g_n(x)\,, \tag{9}$$

where $g_n(x) = p(x) + p_n(x)$ and

$$p_n(x) = \int_0^1 K(x,y) \sin ny dy.$$

By the Lebesque lemma

$$\lim_{n\to\infty} \|p_n\|_{C[a,b]} = 0.$$

However, it is clear that

$$\|f_n(x) - f(x)\|_{C[0,1]} = \|\sin nx\|_{C[0,1]}$$

is not small for large n.

Tikhonov's theorem

Theorem (Tikhonov, 1943). *Let B_1 and B_2 be two Banach spaces. Let $U \subset B_1$ be a compact set and $F : U \to B_2$ be a continuous operator. Assume that the operator F is one-to-one. Let $V = F(U)$. Then the inverse operator $F^{-1} : V \to U$ is continuous.*

Proof. Assume the opposite: that the operator F^{-1} is not continuous on the set V. Then there exists a point $y_0 \in V$ and a number $\varepsilon > 0$ such that for any $\delta > 0$ there exists a point y_δ such that although $\|y_\delta - y_0\|_{B_2} < \delta$, still $\left\|F^{-1}(y_\delta) - F^{-1}(y_0)\right\|_{B_1} \geq \varepsilon$. Hence, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \delta_n = 0^+$ and the corresponding sequence $\{y_n\}_{n=1}^{\infty} \subset V$ such that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n, \| \underbrace{F^{-1}(y_n)}_{x_n} - \underbrace{F^{-1}(y_0)}_{x_0} \|_{B_1} \geq \varepsilon. \qquad (10)$$

Denote

$$x_n = F^{-1}(y_n), x_0 = F^{-1}(y_0). \quad (11)$$

Then by (10) we have

$$\|x_n - x_0\|_{B_1} \geq \varepsilon. \quad (12)$$

Since U is a compact set and all points $x_n \in U$, then one can extract a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_n\}_{n=1}^{\infty}$ from the sequence $\{x_n\}_{n=1}^{\infty}$. Let $\lim_{k\to\infty} x_{n_k} = \overline{x}$. Then $\overline{x} \in U$. Since $F(x_{n_k}) = y_{n_k}$ and the operator F is continuous, then by (10) and (11) we have:

$$\begin{aligned} x_n &= F^{-1}(y_n) \Rightarrow F(x_n) = y_n, \\ x_0 &= F^{-1}(y_0) \Rightarrow F(x_0) = y_0; \\ F(\bar{x}) &= \lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = y_0 \end{aligned} \quad (13)$$

So, we obtained, $F(\bar{x}) = y_0$ since we have that

$$\|y_{\delta_n} - y_0\|_{B_2} < \delta_n$$

But also $F(x_0) = y_0$. Since the operator F is one-to one, we should have $\overline{x} = x_0$. However, by (12) $\|\overline{x} - x_0\|_{B_1} \geq \varepsilon$. We got a contradiction. □

Model inverse problems

We will considers examples of following model inverse problems:

- Elliptic inverse problems
  - Elliptic CIPs
  - Cauchy problem
  - Inverse source problem
  - Inverse spectral problem
- Hyperbolic CIPs
- Parabolic CIPs
- Determination of the initial condition in hyperbolic or parabolic PDE

Cauchy problem

Cauchy problem arises, for example, in electrocardiography and geophysical prospectation. This problem is severally ill-posed and lacks a continuous dependence on data.
Let Γ_c and $\Gamma_i = \Gamma \setminus \Gamma_c$ be two disjoint parts of the boundary Γ. Here,

- Γ_c - observation boundary
- Γ_i - boundary, where observations are not taken

The Cauchy problem reads: given the Cauchy data g and h on the boundary Γ_c, find the function u on the boundary Γ_i, or:

$$
\begin{aligned}
-\nabla \cdot (a(x)\nabla u) &= 0, x \in \Omega, \\
u &= g, \quad x \in \Gamma_c, \\
a\frac{\partial u}{\partial n} &= h, \quad x \in \Gamma_c.
\end{aligned} \tag{14}
$$

Hadamard's example for the Cauchy problem

This example shows that the Cauchy problem for Laplace equation does't depends continuously on data. Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and the boundary $\Gamma_c = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Consider the solution $u = u_n, n = 1, 2, ...$ to the Cauchy problem

$$\begin{aligned} \triangle u &= 0, \quad x \in \Omega, \\ u &= 0, \quad x \in \Gamma_c, \\ \frac{\partial u}{\partial n} &= -n^{-1} \sin n x_1, \quad x \in \Gamma_c. \end{aligned} \tag{15}$$

The function

$$u_n = n^{-2} \sin n x_1 \sinh n x_2 = n^{-2} \sin n x_1 (e^{n x_2} - e^{-n x_2})/2$$

is the solution of the problem (15) and it is a unique solution (by Holmgren's theorem for Laplace equation).

We observe that on Γ_c we have $\lim_{n\to\infty} \frac{\partial u_n}{\partial n} = 0$. However, for all $x_2 > 0$ the solution $\lim_{n\to\infty} u_n(x_1, x_2) = \lim_{n\to\infty} n^{-2} \sin n x_1 \sinh n x_2 = \lim_{n\to\infty} n^{-2} \sin n x_1 (e^{n x_2} - e^{-n x_2})/2 = \infty$.

Inverse source problem

The classical linear inverse problem is to recover a source function f in the equation

$$-\triangle u = f, \quad x \in \Omega \tag{16}$$

from the Cauchy data (g, h) on the boundary Γ:

$$u = g, \quad x \in \Gamma, \tag{17}$$

$$\frac{\partial u}{\partial n} = h, \quad x \in \Gamma. \tag{18}$$

Applications of this problem are in electroencephalography to determine electrical activities of brain from electrodes placed on a head, and electrocardiography to determine heart's electrical activity from body-surface potential distribution. This problem does't has unique solution.
This can be proved if we add one compactly supported function and obtain a different source on the rhs of (16) with the same Cauchy data which is not changed.

Inverse source problem: example of non-uniqueness

Let $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$ is a bounded domain with the boundary Γ. Let $\omega_i, i = 1, 2$ be two balls which have different radius r_i, respectively, centered at the origin o, and these balls are inside the domain Ω. Choose the scalars $\lambda_i : \lambda_1 r_1^d = \lambda_2 r_2^d$. Let the source has the form $f_i = \lambda_i \xi_{\omega_i}$, where ξ denotes the characteristic, or indicator function of the set S in the Laplace equation for $i = 1, 2$

$$-\triangle u_i = f_i, \quad x \in \Omega, \tag{19}$$

$$u_i = g, \quad x \in \Gamma. \tag{20}$$

Inverse source problem: example of non-uniqueness

Then for $\forall v \in H(\Omega) = \{v \in H^2(\Omega) : \triangle v = 0\}$ the variational formulation of (19) will be:

$$
\begin{aligned}
-(\triangle u_i, v)_\Omega &= -[(v, \frac{\partial u_i}{\partial n})_\Gamma - (\nabla u_i, \nabla v)_\Omega] && (21)\\
&= -(v, \frac{\partial u_i}{\partial n})_\Gamma + (u_i, \frac{\partial v}{\partial n})_\Gamma - (u_i, \triangle v)_\Omega && (22)\\
&= (u_i, \frac{\partial v}{\partial n})_\Gamma - (v, \frac{\partial u_i}{\partial n})_\Gamma = (f_i, v), && (23)
\end{aligned}
$$

where $(\cdot, \cdot)$ is the standard L_2 inner product.

Inverse source problem: example of non-uniqueness

Since $u_i = g, i = 1, 2, \quad x \in \Gamma$ then the equation above can be rewritten as

$$(f_i, v)_\Omega = (g, \frac{\partial v}{\partial n})_\Gamma - (v, \frac{\partial u_i}{\partial n})_\Gamma, i = 1, 2. \tag{24}$$

By the mean value theorem for harmonic functions we have

$$(f_i, v)_\Omega = |\omega_i| v(O), \quad i = 1, 2. \tag{25}$$

By construction of f_i and using (24), (25) we get
$\forall v \in H(\Omega) = \{v \in H^2(\Omega) : \triangle v = 0\}$

$$0 = (f_2, v) - (f_1, v) = (v, \frac{\partial u_1}{\partial n})_\Gamma - (v, \frac{\partial u_2}{\partial n})_\Gamma, i = 1, 2. \tag{26}$$

Since $\forall h \in H^{1/2}(\Gamma)$ there exists the harmonic function $v \in H(\Omega)$ such that $v = h$ on Γ and $H^{1/2}$ is dense in $L^2(\Gamma)$ we conclude that $(\frac{\partial u_1}{\partial n})_\Gamma = (\frac{\partial u_2}{\partial n})_\Gamma$ what means that two different sources have identical Cauchy data.

How to solve inverse source problems

- In practical applications it is often requered minimum-norm sources or harmonic sources such that $\triangle f = 0$
- Often are considered localized sources modeled by monopoles, dipoles or their combinations. In the case of combinations of monopoles and dipoles can be obtained unique recovery of the source function via Holmgren's theorem.
- Direct algorithms for location of monopoles and dipoles are developed in

  El Badia, Ha-Duong, An inverse source problem in potential analysis, *Inverse Problems*, 16, pp.651–663, 2000.

Inverse spectral problem

The forward problem is

$$Au = \lambda u, \tag{27}$$

where A is an elliptic operator, λ is eigenvalue and u is respective eigenfunction.
The inverse spectral problem is to recover the coefficients in the operator A or the geometry of the domain Ω from partial or multiple spectral data (knowledge of eigenvalues and eigenfunctions).

Example of an inverse spectral problem

Let the operator A is applied to u as:

$$Au = -u''(t) + q(t)u(t), \quad t \in (0,1) \tag{28}$$

where $q(t)$ is the potential. Then the classical **Sturm-Liouville problem** reads: given a potential $q(t)$ and constants $h, H > 0$ find eigenvalues $\{\lambda_k\}$ and eigenfunctions $\{u_k\}$ such that

$$-u''(t) + q(t)u(t) = \lambda u(t), t \in (0,1), \tag{29}$$

$$u'(0) - hu(0) = 0, \tag{30}$$

$$u'(1) + Hu(1) = 0. \tag{31}$$

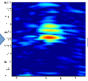
The set of eigenvalues $\{\lambda_k\}$ is real and countable.
The **inverse Sturm-Liouville problem** is to recover the potential $q(t), h, H$ from the knowledge of spectral data $(\{\lambda_k\}, \{u_k\})$. This data can take different forms.
Numerical solution of these problems is presented in
M. T. Chu, G. H. Golub, *Inverse eigenvalue problems*, Oxford University Press, Oxford, New York, 2005.

Coefficient Inverse Problems: main steps in solution

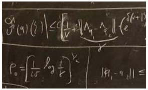
Coefficient Inverse Problems for PDE

A coefficient inverse problem for a given partial differential equation (PDE) aims at estimating a spatially distributed coefficient of the model PDE using measurements taken on the boundary of the domain of interest.

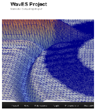
Use collected data. Data is generated by different type of waves (acoustic, elastic, electromagnetic)



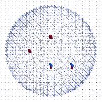
Extract image data for analysis.



Develop new mathematical methods for reconstruction of spatially distributed coefficients in the model PDE which describes the physical process.



Implement new methods using fast and robust numerical methods.



Visualize and analyze obtained reconstructions. Estimate error in the reconstruction.

The Hyperbolic Coefficient Inverse Problem

Let us assume that the domain Ω is a ball,
$\Omega = \{|x| < R\} \subset \mathbb{R}^n, R = const. > 0$. Let $T = const. > 0$. Denote
$Q_T^{\pm} = \Omega \times (-T, T)\,, S_T^{\pm} = \partial\Omega \times (-T, T)$.
Let the function $u(x,t) \in C^2\left(\overline{Q}_T\right)$ satisfies to the

$$c\,(x) u_{tt} = \Delta u + \sum_{|\alpha| \leq 1} a_{\alpha}\,(x) D_x^{\alpha} u, \text{ in } Q_T, \tag{32}$$

$$u\,(x,0) = f_0\,(x), \quad u_t\,(x,0) = f_1\,(x), \tag{33}$$

$$u|_{S_T} = p\,(x,t), \quad \frac{\partial u}{\partial n}|_{S_T} = q\,(x,t), \tag{34}$$

where functions $a_{\alpha}, c \in C\left(\overline{Q}_T\right)$ and $c \geq 1$.
The Hyperbolic Coefficient Inverse Problem. Suppose that one of coefficients in equation (32) is unknown inside of the ball Ω and is known outside of it. Assume that all other coefficients in (35) are known and conditions (33) are satisfied. Determine that unknown coefficient inside of Ω, assuming that the functions $p\,(x,t)$ and $q\,(x,t)$ in (34) are known.

The Coefficient Inverse Problem for a parabolic equation

Consider the Cauchy problem for the following forward parabolic equation

$$c(x)u_t = \Delta u + \sum_{|\alpha|\leq 1} a_\alpha(x) D_x^\alpha u, \text{ in } D_T^{n+1} = \mathbb{R}^n \times (0, T), \tag{35}$$

$$u(x,0) = f_0(x), \tag{36}$$

$$c, a_\alpha \in C^\beta(\mathbb{R}^n), f_0 \in C^{2+\beta}(\mathbb{R}^n), \beta \in (0,1), c(x) \geq 1. \tag{37}$$

Given conditions (37), this problem has unique solution $u \in C^{2+\beta,1+\beta/2}\left(\overline{D}_T^{n+1}\right)$. Assume that $\Omega = \{|x| < R\} \subset \mathbb{R}^n, n \geq 2$. Let $\Gamma \subseteq \partial\Omega$ be a part of the boundary of the domain Ω and $T = const. > 0$.
The Parabolic Coefficient Inverse Problem. Suppose that one of coefficients in equation (35) is unknown inside of the ball Ω and is known outside of it. Assume that all other coefficients in (35) are known and conditions (36), (37) are satisfied. Determine that unknown coefficient inside of Ω, assuming that the following functions $p(x,t)$ and $q(x,t)$ are known

$$u\mid_{\Gamma_T} = p(x,t), \quad \frac{\partial u}{\partial n}\mid_{\Gamma_T} = q(x,t). \tag{38}$$

CIP for a parabolic equation is an ill-posed problem

Let the function $a(x) \in C^{\alpha}(\mathbb{R}^n), \alpha \in (0,1)$ and $a(x) = 0$ outside of the bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^3$. Consider the following Cauchy problem

$$u_t = \Delta u + a(x)u, \quad (x,t) \in D_T^{n+1}, \tag{39}$$

$$u(x,0) = f(x). \tag{40}$$

Here the function $f(x) \in C^{2+\alpha}(\mathbb{R}^n)$ has a finite support in $\mathbb{R}^n$. Another option for the initial condition is

$$f(x) = \delta(x - x_0), \quad x_0 \notin \overline{\Omega} \tag{41}$$

The inverse problem is: assume that the function $a(x)$ is unknown inside of the domain Ω. Determine this function for $x \in \Omega$ assuming that the following function $g(x,t)$ is known

$$u\,|_{S_T} = g(x,t). \tag{42}$$

CIP for a parabolic equation is an ill-posed problem

Let us show that this CIP is an ill-posed problem. Let the function u_0 be the fundamental solution of the heat equation $u_{0t} = \Delta u_0$,

$$u_0(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n} \exp\left(-\frac{|x|^2}{4t}\right).$$

It is well known that by [LSU]

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-\xi,t) f(\xi)\, d\xi + \int_0^t \int_\Omega u_0(x-\xi,t-\tau)\, a(\xi)\, u(\xi,\tau)\, d\tau. \tag{43}$$

Because of the presence of the integral $\int_0^t (\cdot)\, d\tau$ the integral (43) is a Volterra-like integral equation of the second kind.

[LSU] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, R.I., 1968.

Hence, it can be solved as in [LSU]

$$u(x,t) = \underbrace{\int\limits_{\mathbb{R}^n} u_0(x-\xi,t)\, f(\xi)\, d\xi}_{u_0^f} + \sum_{n=1}^{\infty} u_n(x,t), \qquad (44)$$

$$u_n(x,t) = \int\limits_0^t \int\limits_{\Omega} u_0(x-\xi,t-\tau)\, a(\xi)\, u_{n-1}(\xi,\tau)\, d\tau.$$

One can prove that each function $u_n \in C^{2+\alpha,1+\alpha/2}\left(\overline{D}_T^{n+1}\right)$ and using [LSU]

$$\left| D_x^{\beta} D_t^{k} u_n(x,t) \right| \leq \frac{(Mt)^n}{n!}, \quad |\beta| + 2k \leq 2, \qquad (45)$$

where $M = \|a\|_{C^{\alpha}(\overline{\Omega})}$. In the case when $f = \delta(x-x_0)$ the first term in the right hand side of (44) should be replaced with $u_0(x-x_0,t)$.

Let $u_0^f(x,t)$ be the first term of the right hand side of (44) and $v(x,t) = u(x,t) - u_0^f(x,t)$.
Using (45), one can rewrite (44) as

$$v(x,t) = \int_0^t \int_\Omega u_0(x-\xi, t-\tau)\left(a(\xi)\, u_0^f(\xi,\tau) + P(a)(\xi,\tau)\right) d\xi d\tau, \tag{46}$$

where $P(a)$ is a nonlinear operator applied to the function a. It is clear from (44)-(46) that the operator $P : C^{\alpha}\left(\overline{\Omega}\right) \to C^{2+\alpha,1+\alpha/2}\left(\overline{Q}_T\right)$ is continuous. Setting in (46) $(x,t) \in S_T$, recalling (42) and denoting denoting $\overline{g}(x,t) = g(x,t) - u_0^f(x,t)$, we obtain a nonlinear integral equation of the first kind with respect to the unknown coefficient $a(x)$

$$\underbrace{\int_{S_T} u_0(x-\xi, t-\tau)\left(u_0^f(\xi,\tau)\, a(\xi) + P(a)(\xi,\tau)\right) d\xi d\tau}_{A(a)} = \overline{g}(x,t), \quad (x,t) \in S_T. \tag{47}$$

Let $A(a)$ be the operator in the left hand side of (47). Let $H_1 = L_2(\Omega)$ and $H_2 = L_2(S_T)$. Consider now the set U of functions defined as

$$U = \left\{ a : a \in C^{\alpha}\left(\overline{\Omega}\right), \|a\|_{C^{\alpha}\left(\overline{\Omega}\right)} \leq M \right\} \subset H_1.$$

Since the $L_2(\Omega)$ −norm is weaker than the $C^{\alpha}\left(\overline{\Omega}\right)$ −norm, then U is a bounded set in H_1 and $A : U \to C(S_T)$ is a compact operator by Theorem 1.1 of [BK]. Since the norm in $L_2(S_T)$ is weaker than the norm in $C(S_T)$, then $A : U \to H_2$ is also a compact operator. However, U is not a compact. Hence, from the Theorem about an ill-posed problem (Theorem 1.2 of [BK]) follows that the problem of solution of the equation

$$A(a) = g, a \in U \subset H_1, g \in H_2$$

is an ill-posed problem.

Determination of initial condition in parabolic PDE

- The inverse heat conduction problem has applications in image processing, remote sensing, oil base detection etc.
- Consider the direct Cauchy problem for the following parabolic equation

$$\begin{aligned} u_t &= C^2(x)\Delta u, \\ u(\cdot, 0) &= f(x). \end{aligned} \tag{48}$$

  If (48) is considered in $\Omega \times (0, T]$ and supplied by appropriate boundary conditions, then with known $f(x) \in C^2(\mathbb{R}^n)$, boundary conditions and $C(x) \in C(\mathbb{R}^n)$ the problem can be solved.
- **The inverse problem is**: determine the initial condition $f(x)$ in (48) by knowing the function u at the final time T:

$$u(\cdot, T) = g(x). \tag{49}$$

Determination of initial condition in parabolic PDE

Two types of algorithms can be considered to solve the inverse problem:

- Optimization approach and construction of the adjoint problem to be used in the iterative gradient update. Read about this approach in

  Dinh Nho Hào Nguyen Thi Ngoc Oanh, Determination of the initial condition in parabolic equations from integral observations, *Inverse Problems in Science and Engineering*, 25:8, 1138-1167, 2017. DOI: 10.1080/17415977.2016.1229778

- Algorithm for solution of the inverse problem which is based on the reduction of solution of (48) to the solution of Fredholm integral equation of the first kind by the method of separation of variables and then obtaining of the initial condition $f(x)$ by the inverse Fourier transform of this solution.See more info in

  Tao Min, Bei Geng, Jucheng Ren, Inverse estimation of the initial condition for the heat equation, *International Journal of Pure and Applied Mathematics*, 82(4), 581-593, 2013.

Acoustic CIPs

- Acoustic CIPs for acoustic wave equation

$$\frac{1}{c^2(x)} u_{tt} = \Delta u \text{ in } \mathbb{R}^3 \times (0, \infty), \tag{50}$$

$$u(x,0) = 0, u_t(x,0) = \delta(x - x_0). \tag{51}$$

- Let now introduce the convex bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary $\partial\Omega \in C^3$ and specify time variable $t \in [0, T]$. Next, we supply the Cauchy problem by the appropriate b.c.
- We assume that the coefficient $c(x)$ belongs to the set of admissible parameters M which should be specified for the concrete problem.
- $u(x,t)$ acoustic pressure - we measure it on the boundary $\partial\Omega$.
- $c(x)$ speed of sound – want to determine by measured $u(x,t)$ on the boundary $\partial\Omega$
- Applications: medical imaging, electromagnetic, acoustics, geological profiling, construction of new materials

Acoustic CIP: example

We model the process of electric wave field propagation in non-conductive and nonmagnetic media with $\nabla \cdot (\varepsilon E) = 0$ via a single hyperbolic PDE, which is the same as an acoustic wave equation (50)-(51). The forward problem is the following Cauchy problem

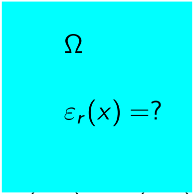
$$\begin{aligned} \varepsilon_r(x) u_{tt} &= \Delta u, \text{ in } \mathbb{R}^3 \times (0, \infty), && (52) \\ u(x, 0) &= 0, \; u_t(x, 0) = \delta(x - x_0). && (53) \end{aligned}$$

Here, $\varepsilon_r(x)$ is the spatially distributed dielectric constant (relative dielectric permittivity),

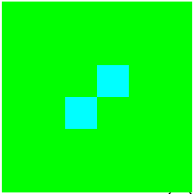
$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0}, \; \sqrt{\varepsilon_r(x)} = n(x) = \frac{c_0}{c(x)} \geq 1, \qquad (54)$$

where ε_0 is the dielectric permittivity of the vacuum (which we assume to be the same as the one in the air), $\varepsilon(x)$ is the dielectric permittivity of the medium of interest, $n(x)$ is the refractive index of the medium of interest, $c(x)$ is the speed of the propagation of the EM field in this medium, and c_0 is the speed of light in the vacuum, which we assume to be the same as one in the air.

Acoustic CIP: example



$u(x,t) = g(x,t)$ on $\partial\Omega$

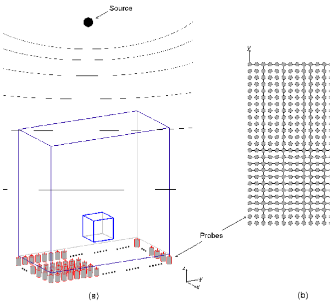


exact $\varepsilon_r(x)$

Coefficient Inverse Problem *Assume that the function* $\varepsilon_r(x)$ *is unknown in the domain* Ω. *Determine the function* $\varepsilon_r(x)$ *for* $x \in \Omega$, *assuming that the following function* $g(x,t)$ *is known for a source* $x_0 \notin \overline{\Omega}$

$$u(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,T].$$

Reconstruction of dielectrics from experimental data



a) The rectangular prism depicts our computational domain Ω. Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved EM wave were conducted on the bottom side of this prism. The signal was measured with the time interval 20 picoseconds with total time 12.3 nanoseconds. b) Schematic diagram of locations of detectors on the bottom side of the prism Ω. The distance between neighboring detectors was 10 mm.

L.Beilina, M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally

The two-stage numerical procedure for solution of CIP

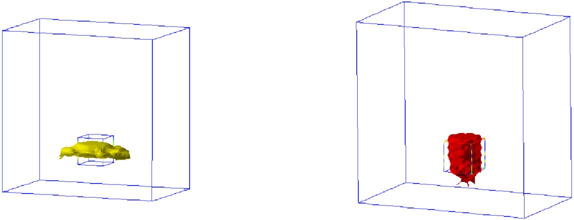
Stage 1. Approximately globally convergent numerical method provides a good approximation for the exact solution ε_{glob}.
Stage 2. Adaptive Finite Element Method refines it via minimization of the corresponding Tikhonov functional with $\varepsilon_0 = \varepsilon_{glob}$:

$$J(u, \varepsilon) = \frac{1}{2} \int_{\Gamma} \int_{0}^{T} (u - \tilde{u})^2 z_{\delta}(t) ds dt + \frac{1}{2} \gamma \int_{\Omega} (\varepsilon - \varepsilon_0)^2 \ \ dx. \qquad (55)$$

where $\tilde{u}$ is the observed wave field in the model PDE (for example, acoustic wave equation), u satisfies this model PDE and thus depends on ε.

The two-stage numerical procedure

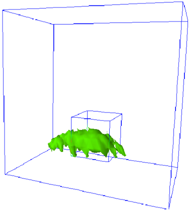
Stage 1. Approximately globally convergent numerical method provides a good approximation for the exact solution.
Stage 2. Adaptive Finite Element Method refines it.



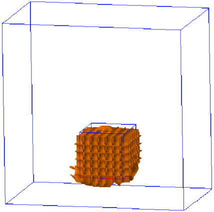
a) $\varepsilon_{\boldsymbol{r}}^{(\mathbf{5},\mathbf{2})} = 3.9$, $n^{(\mathbf{5},\mathbf{2})} = 1.97$ b) $\varepsilon_{\boldsymbol{r},\boldsymbol{h}} \approx 4.2$, $n_{\boldsymbol{glob}} = \sqrt{\varepsilon_{\boldsymbol{r},\boldsymbol{h}}} \approx 2.05$

a) A sample of the reconstruction result of the dielectric cube No. 1 (4 cm side) via the first stage. b) Result after applying the adaptive stage (2-nd stage). The side of the cube is 4 cm=1.33 wavelength.

Results of the two-stage procedure, cube nr.2 (big)



a) $\varepsilon_{\mathbf{r}}(5,5) = 3.19$, $n^{(\mathbf{5},\mathbf{5})} = 1.79$



b) $\varepsilon_{\mathbf{r},\mathbf{h}} \approx 3.0$, $n_{\mathbf{glob}} = \sqrt{\varepsilon_{\mathbf{r},\mathbf{h}}} \approx 1.73$

a) Reconstruction of the dielectric cube No. 2 (6 cm side) via the first stage. b) The final reconstruction result after applying the adaptive stage (2-nd stage). The side 6 cm=2 wavelength.

L.Beilina, M.V.Klibanov, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive inverse algorithm, *Inverse Problems*, 26, 125009, 2010.

Elastic CIPs

Let $v\left(x,t\right)=\left(v_1,v_2,v_3\right)\left(x,t\right)$ be vector of displacement. We consider the Cauchy problem for the elastodynamics equations in the isotropic case in the entire space $\mathbb{R}^3$,

$$\begin{aligned} \rho(x)\frac{\partial^2 v}{\partial t^2}-\nabla\cdot\tau=\delta(x_3-z_0)f(t), \\ \tau=C\epsilon, \\ v\left(x,0\right)=0, v_t(x,0)=0, \quad x\in\mathbb{R}^3, t\in\left(0,T\right), \end{aligned} \tag{56}$$

where $v(x,t)$ is the total displacement generated by the incident plane wave $f(t)$ propagating along the x_3-axis which is incident at the plane $x_3=z_0$, $\rho(x)$ is the density of the material, τ is the stress tensor, C is a cyclic symmetric tensor and ϵ is the strain tensor which have components

$$\epsilon_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right), \quad i,j=1,2,3.$$

The strain tensor ϵ is coupled with the stress tensor τ by the Hooke's law

$$\tau_{i,j}=\sum^3\sum^3 C_{ijkl}\epsilon_{kl} \tag{57}$$

Elastic CIPs

C is a cyclic symmetric tensor

$$C_{ijkl} = C_{klij} = C_{jkli}.$$

When C_{ijkl} does not depends on $\mathbf{x}$ then material of the domain which we consider is said to be homogeneous. If the tensor C_{ijkl} does not depends on the choice of the coordinate system, then the material of the domain under interest is said to be isotropic. Otherwise, the material is anisotropic.

In the isotropic case the cyclic symmetric tensor C can be written as

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}),$$

where δ_{ij} is the Kronecker delta, in which case the equation (57) takes the form

$$\tau_{i,j} = \lambda \delta_{ij} \sum_{k=1}^{3} \epsilon_{kk} + 2\mu\epsilon_{ij}, \tag{58}$$

where λ and μ are Lame coefficients.

Elastic CIPs

Lame coefficients λ and μ are given by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{59}$$

Here, E is the modulus of elasticity, or Young modulus, and ν is the Poisson's ratio of the elastic material. Following relations should be satisfied

$$\lambda > 0, \mu > 0 \iff E > 0, 0 < \nu < 1/2. \tag{60}$$

Elastic CIPs

To write the equation (56) only in terms of v we eliminate the strain tensor ϵ from (56) using (57). Then in the isotropic case the equation in (56) writes

$$
\begin{aligned}
&\rho(x)\frac{\partial^2 v_1}{\partial t^2} - \frac{\partial}{\partial x_1}\left((\lambda+2\mu)\frac{\partial v_1}{\partial x_1} + \lambda\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_3}{\partial x_3}\right) \\
&\quad - \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)\right) = 0, \\
&\rho(x)\frac{\partial^2 v_2}{\partial t^2} - \frac{\partial}{\partial x_2}\left((\lambda+2\mu)\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_1}{\partial x_1} + \lambda\frac{\partial v_3}{\partial x_3}\right) \\
&\quad - \frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\right) - \frac{\partial}{\partial x_3}\left(\mu\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right)\right) = 0, \\
&\rho(x)\frac{\partial^2 v_3}{\partial t^2} - \frac{\partial}{\partial x_3}\left((\lambda+2\mu)\frac{\partial v_3}{\partial x_3} + \lambda\frac{\partial v_2}{\partial x_2} + \lambda\frac{\partial v_1}{\partial x_1}\right) \\
&\quad - \frac{\partial}{\partial x_2}\left(\mu\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right)\right) - \frac{\partial}{\partial x_1}\left(\mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)\right) = \delta(x_3 - z_0)f(t),
\end{aligned}
\tag{61}
$$

Elastic CIPs

The system above for $\lambda = const. > 0, \mu = const. > 0$ can be written in a more compact form as

$$\rho \frac{\partial^2 v}{\partial t^2} - \mu \nabla \cdot (\nabla v) - (\lambda + \mu) \nabla (\nabla \cdot v) = \delta(x_3 - z_0) f(t),$$
$$v(x, 0) = 0, v_t(x, 0) = 0, \quad x \in \mathbb{R}^3, t \in (0, T). \tag{62}$$

Inserting Helmholtz decomposition

$$v = \nabla \varphi + \nabla \times \psi \tag{63}$$

with a scalar potential φ and a vector potential ψ into (62) we get

$$\rho \frac{\partial^2 (\nabla \varphi + \nabla \times \psi)}{\partial t^2} - \mu \nabla \cdot (\nabla (\nabla \varphi + \nabla \times \psi)) - (\lambda + \mu) \nabla (\nabla \cdot (\nabla \varphi + \nabla \times \psi))$$
$$= \delta(x_3 - z_0) f(t) \tag{64}$$

Elastic CIPs

Using

$$\nabla \cdot (\nabla \varphi) = \triangle \varphi,$$
$$\nabla \cdot (\nabla \times \psi) = 0,$$

we finally get with $f(t) = 0$

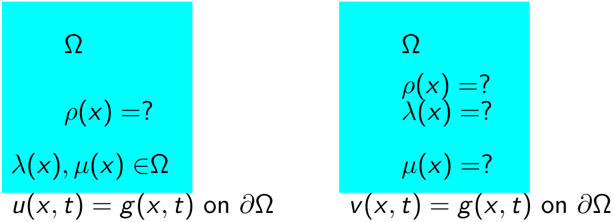
$$\nabla \left(\rho \frac{\partial^2 \varphi}{\partial t^2} - (\lambda + 2\mu) \triangle \varphi \right) + \nabla \times \left(\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \triangle \psi \right) = 0 \qquad (65)$$

We conclude that

$$\rho \frac{\partial^2 \varphi}{\partial t^2} - (\lambda + 2\mu) \triangle \varphi = 0,$$
$$\rho \frac{\partial^2 \psi}{\partial t^2} - \mu \triangle \psi = 0. \qquad (66)$$

Here, $v = \nabla \varphi$ is the pressure wave with the spead $V_p = (\frac{\lambda + 2\mu}{\rho})^{1/2}$, $v = \nabla \times \psi$ is the shear wave with the spead $V_s = (\frac{\mu}{\rho})^{1/2}$

Elastic CIP: examples of CIPs



Inverse Problem (IP1) *Determine the density function $\rho(x)$ in Ω for $x \in \Omega$ assuming that the Lame parameters $\lambda(x), \mu(x)$ and $g(x,t)$ s.t.*

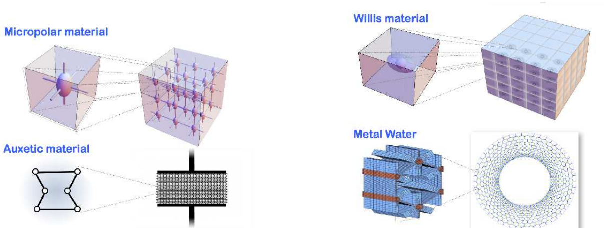
$$v(x,t) = g(x,t), \forall (x,t) \in \partial\Omega \times (0,T].$$

are known in Ω.

Inverse Problem (IP2) *Determine the functions $\rho(x), \lambda(x), \mu(x)$ in Ω for $x \in \Omega$ assuming that $g(x,t)$ is known in Ω.*

Applications of elastic CIP: design of new materials

There is a class of materials for which the macroscale properties can be obtained more from such called mechanical *microstructural* design, see Figure for examples of such materials.



- Practical applications: mechanical cloaking, control and manipulation of waves in fluids and solids, etc.
- Examples of such materials include nanomaterials such as graphene or carbon nanotubes with extraordinary strength properties.
- Design of new mechanical metamaterials using computational modeling is one of the applications of elastic CIPs.

CIP for electromagnetic problems. Maxwell's equations

Consider a region of space that has no electric or magnetic current sources, but may have materials that absorb electric or magnetic field energy. Then, using MKS units, the time-dependent Maxwell's equations are given in differential and integral form by *Faraday's law* :

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{M} \tag{67a}$$

$$\frac{\partial}{\partial t} \iint_A \mathbf{B} \cdot \mathbf{dA} = -\oint_L \mathbf{E} \cdot \mathbf{dL} - \iint_A \mathbf{M} \cdot \mathbf{dA} \tag{67b}$$

The MKS system of units is a physical system of units that expresses any given measurement using fundamental units of the metre, kilogram, and/or second (MKS))

A. Taflove, S. C. Hagness, Computational Electromagnetics. The finite-difference time-domain method, 3rd edition, *Artech House Publishers*, 2005.

Maxwell's equations

Ampere's law :

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J} \tag{68a}$$

$$\frac{\partial}{\partial t} \iint\limits_A \mathbf{D} \cdot \mathbf{dA} = \oint\limits_L \mathbf{H} \cdot \mathbf{dL} - \iint\limits_A \mathbf{J} \cdot \mathbf{dA} \tag{68b}$$

Gauss' law for the electric field :

$$\nabla \cdot \mathbf{D} = 0 \tag{69a}$$

$$\oiint\limits_A \mathbf{D} \cdot \mathbf{dA} = 0 \tag{69b}$$

Gauss' law for the magnetic field :

$$\nabla \cdot \mathbf{B} = 0 \tag{70a}$$

$$\oiint\limits_A \mathbf{B} \cdot \mathbf{dA} = 0 \tag{70b}$$

Maxwell's equations

In (67) to (70), the following symbols (and their MKS units) are defined:

$\mathbf{E}$	:	electric field (volts/meter)
$\mathbf{D}$	:	electric flux density (coulombs/meter^2)
$\mathbf{H}$	:	magnetic field (amperes/meter)
$\mathbf{B}$	:	magnetic flux density (webers/meter^2)
A	:	arbitrary three-dimensional surface
$\mathbf{dA}$	:	differential normal vector that characterizes surface A (meter^2)
L	:	closed contour that bounds surface A (volts/meter)
$\mathbf{dL}$	:	differential length vector that characterizes contour L (meters)
$\mathbf{J}$	:	electric current density (amperes/meter^2)
$\mathbf{M}$	:	equivalent magnetic current density (volts/meter^2)

Maxwell's equations

In linear, isotropic, nondispersive materials (i.e. materials having field-independent, direction-independent, and frequency-independent electric and magnetic properties), we can relate **D** to **E** and **B** to **H** using simple proportions:

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E}; \quad \mathbf{B} = \mu \mathbf{H} = \mu_r \mu_0 \mathbf{H} \tag{71}$$

where

ε	:	electrical permittivity (farads/meter)
ε_r	:	relative permittivity (dimensionless scalar)
ε_0	:	free-space permittivity (8.854×10^{-12} farads/meter)
μ	:	magnetic permeability (henrys/meter)
μ_r	:	relative permeability (dimensionless scalar)
μ_0	:	free-space permeability ($4\pi \times 10^{-7}$ henrys/meter)

Note that **J** and **M** can act as *independent sources* of E- and H-field energy, $\mathbf{J}_{source}$ and $\mathbf{M}_{source}$.

Maxwell's equations

We also allow for materials with isotropic, nondispersive electric and magnetic losses that attenuate E- and H-fields via conversion to heat energy. This yields

$$\mathbf{J} = \mathbf{J}_{source} + \sigma \mathbf{E}; \quad \mathbf{M} = \mathbf{M}_{source} + \sigma^* \mathbf{H} \tag{72}$$

where σ : electric conductivity (siemens/meter)
σ^* : equivalent magnetic loss (ohms/meter)

Finally, we substitute (71) and (72) into (67a) and (68a). This yields Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu}\nabla \times \mathbf{E} - \frac{1}{\mu}\left(\mathbf{M}_{source} + \sigma^* \mathbf{H}\right) \tag{73}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon}\nabla \times \mathbf{H} - \frac{1}{\varepsilon}\left(\mathbf{J}_{source} + \sigma \mathbf{E}\right) \tag{74}$$

Maxwell's equations

We now write out the vector components of the curl operators of (73) and (74) in Cartesian coordinates. This yields the following system:

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \left(M_{source_{\mathbf{x}}} + \sigma^* H_x\right)\right] \tag{75a}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \left(M_{source_{\mathbf{y}}} + \sigma^* H_y\right)\right] \tag{75b}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \left(M_{source_{\mathbf{z}}} + \sigma^* H_z\right)\right] \tag{75c}$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \left(J_{source_{\mathbf{x}}} + \sigma E_x\right)\right] \tag{76a}$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \left(J_{source_{\mathbf{y}}} + \sigma E_y\right)\right] \tag{76b}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \left(J_{source_{\mathbf{z}}} + \sigma E_z\right)\right] \tag{76c}$$

Fourier transform

To formulate forward problem in electrical prospecting we use definition of the Fourier transform.

- Recall that if $f(x)$ is an integrable function in **R** then its Fourier transform is the function $\hat{f}(\xi)$ on **R** such that

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} \exp^{-i\xi x} f(x)\, dx \tag{77}$$

- To recover function $f(x)$ from $\hat{f}(\xi)$ is used inverse Fourier transform which is given by the formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp^{i\xi x} \hat{f}(\xi)\, d\xi \tag{78}$$

Maxwell's equations in electrical prospecting

- CIPs of electrical prospecting appears in subsurface imaging. The CIP is as follows: the electromagnetic field is measured on the surface of the ground. The problem is to find the electric conductivity σ and magnetic permeability μ of the geological medium.
- We use definition of the Fourier transform for the functions $E(x,t)$ and $H(x,t)$. If $E(x,t)$ and $H(x,t)$ are integrable functions in $\mathbf{R}^3 \times (-\infty, +\infty)$ then we define their Fourier transforms as the functions $E(x,\omega)$ and $H(x,\omega)$ on $\mathbf{R}^3$ such that

$$\begin{aligned} E(x,\omega) &= \int_{-\infty}^{+\infty} E(x,t) \exp^{-i\omega t} dt, \\ H(x,\omega) &= \int_{-\infty}^{+\infty} H(x,t) \exp^{-i\omega t} dt \end{aligned} \tag{79}$$

V. P. Gubatenko, On the formulation of Inverse Problem in electrical prospecting, Inverse Problems and Large-Scale Computations, *Springer Proceedings in Mathematics and Statistics*, 52, 2013.

Maxwell's equations in electrical prospecting

Now, we apply (79) to the system (73)-(74). We multiply system (73)-(74) with $\exp^{-i\omega t}$ and integrate it in time to get

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t}\, dt &= -\frac{1}{\mu} \int_{-\infty}^{+\infty} \nabla \times \mathbf{E} \exp^{-i\omega t}\, dt \\
&- \frac{1}{\mu}\Big(\int_{-\infty}^{+\infty} \mathbf{M}_{source} \exp^{-i\omega t}\, dt + \sigma^* \int_{-\infty}^{+\infty} \mathbf{H} \exp^{-i\omega t}\, dt \Big) \\
\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t}\, dt &= \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \nabla \times \mathbf{H} \exp^{-i\omega t}\, dt \\
&- \frac{1}{\varepsilon}\Big(\int_{-\infty}^{+\infty} \mathbf{J}_{source} \exp^{-i\omega t}\, dt + \sigma \int_{-\infty}^{+\infty} \mathbf{E} \exp^{-i\omega t}\, dt \Big)
\end{aligned} \tag{80}$$

In this system we consider $\mathbf{M}_{source} = 0, \sigma^* = 0, \mathbf{J}_{source} = 0$ in accordance with applications in electrical prospecting.

Maxwell's equations in electrical prospecting

In this (80) we consider $\mathbf{M}_{source} = 0, \sigma^* = 0, \mathbf{J}_{source} = 0$ in accordance with applications in electrical prospecting such that the above system is reduced to the system

$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t} \, dt = -\frac{1}{\mu} \int_{-\infty}^{+\infty} \nabla \times \mathbf{E} \exp^{-i\omega t} \, dt$$
$$\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t} \, dt = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \nabla \times \mathbf{H} \exp^{-i\omega t} \, dt - \frac{1}{\varepsilon}\sigma \int_{-\infty}^{+\infty} \mathbf{E} \exp^{-i\omega t} \, dt \tag{81}$$

Maxwell's equations in electrical prospecting

Next, we integrate by parts in time integrals $\int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t}\, dt$ and $\int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t}\, dt$ to obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\partial \mathbf{H}}{\partial t} \exp^{-i\omega t}\, dt &= \exp^{-i\omega t} \mathbf{H}|_{-\infty}^{+\infty} \\ &\quad + i\omega \int_{-\infty}^{+\infty} \mathbf{H} \exp^{-i\omega t}\, dt = i\omega \mathbf{H}(x,\omega) \\ \int_{-\infty}^{+\infty} \frac{\partial \mathbf{E}}{\partial t} \exp^{-i\omega t}\, dt &= \exp^{-i\omega t} \mathbf{E}|_{-\infty}^{+\infty} \\ &\quad + i\omega \int_{-\infty}^{+\infty} \mathbf{E} \exp^{-i\omega t}\, dt = i\omega \mathbf{E}(x,\omega) \end{aligned} \tag{82}$$

and substitute them into (81) to obtain

Maxwell's equations in electrical prospecting

$$\begin{aligned} i\omega\mu\ \mathbf{H}(x,\omega) &= -\nabla \times \mathbf{E}(x,\omega) \\ i\omega\varepsilon\ \mathbf{E}(x,\omega) &= \nabla \times \mathbf{H}(x,\omega) - \sigma\mathbf{E}(x,\omega) \end{aligned} \tag{83}$$

The above system can be rewritten as

$$\begin{aligned} \nabla \times \mathbf{E}(x,\omega) &= -i\omega\mu\ \mathbf{H}(x,\omega) \\ \nabla \times \mathbf{H}(x,\omega) &= (i\omega\varepsilon + \sigma)\mathbf{E}(x,\omega) \end{aligned} \tag{84}$$

According to our applications we assume that $\mu = const.$, $\varepsilon = const. > 0$. We introduce new variable $\sigma_\omega := i\omega\varepsilon + \sigma$ to obtain

$$\begin{aligned} \nabla \times \mathbf{E}(x,\omega) &= -i\omega\mu\ \mathbf{H}(x,\omega) \\ \nabla \times \mathbf{H}(x,\omega) &= \sigma_\omega\mathbf{E}(x,\omega) \end{aligned} \tag{85}$$

CIP in electrical prospecting

Taking operator of $\nabla\times$ from the first equation in system (85) we have

$$\nabla \times \nabla \times \mathbf{E}(x,\omega) = -i\omega\mu \, \nabla \times \mathbf{H}(x,\omega) \tag{86}$$

Substituting the second equation of the system (85) in the right hand side of (86) we obtain

$$\nabla \times \nabla \times \mathbf{E}(x,\omega) = -i\omega\mu \, \sigma_\omega \mathbf{E}(x,\omega) \tag{87}$$

Coefficient Inverse Problem

Let the function $\sigma_\omega(x) \in C^1(\mathbf{R}^3), x \in \mathbf{R}^3$. Let $\Omega \subset \mathbf{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. Determine the coefficient $\sigma_\omega(x) \in \Omega$ assuming that the following function $g(x,\omega)$ is known

$$\mathbf{E}(x,\omega)|_{\partial\Omega} = g(x,\omega) \quad \forall (x,\omega) \in \partial\Omega \times (-\infty,+\infty) \tag{88}$$

CIPs for electric wave propagation

Recall Maxwell's curl equations in linear, isotropic, nondispersive, lossy materials with $\sigma^* = 0, \mathbf{M}_{source} = 0$:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} \quad (89)$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \sigma \mathbf{E} - \frac{1}{\varepsilon} \mathbf{J}_{source} \quad (90)$$

Take now $\frac{\partial}{\partial t}$ from (90) and $\nabla \times$ from (89) to get:

$$\nabla \times \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} \quad (91)$$

$$\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial}{\partial t} \nabla \times \mathbf{H} - \sigma \frac{\partial}{\partial t} \mathbf{E} - \frac{\partial}{\partial t} \mathbf{J}_{source} \quad (92)$$

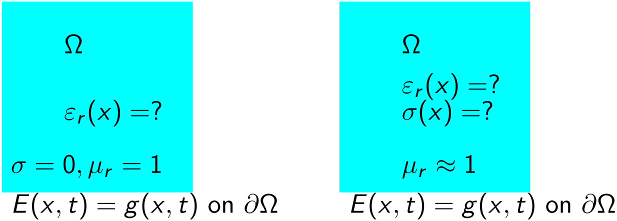
CIPs for electric wave propagation

Substitude the right hand side of (91) into (92) instead of $\frac{\partial}{\partial t}\nabla \times \mathbf{H}$ to obtain Maxwell's equations for electric field $\mathbf{E} = (E_1, E_2, E_3)$. Let us consider now Cauchy problem for the Maxwell's equations for electric field $\mathbf{E}$ in the domain $\Omega_T = \Omega \times [0, T]$:

$$\begin{aligned} \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} = -\sigma \frac{\partial}{\partial t} \mathbf{E} - \frac{\partial}{\partial t} \mathbf{J}_{source} \text{ in } \Omega_T, \\ \nabla \cdot (\varepsilon \mathbf{E}) = 0, \\ \mathbf{E}(\mathbf{x}, \mathbf{0}) = f_0(x), \quad \mathbf{E_t}(\mathbf{x}, \mathbf{0}) = f_1(x) \text{ in } \Omega, \end{aligned} \tag{93}$$

- Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$ and specify time variable $t \in [0, T]$. Next, we supply the Cauchy problem by the appropriate b.c.
- $\varepsilon(x)$ and $\sigma(x)$ are dielectric permittivity and electric conductivity functions, respectively of the domain Ω. In (93), $\varepsilon(x) = \varepsilon_r(x)\varepsilon_0, \mu = \mu_r \mu_0$ and $\sigma(x)$ are dielectric permittivity, permeability and electric conductivity functions, respectively, ε_0, μ_0 are dielectric permittivity and permeability of free space,

CIPs for electric wave propagation



Inverse Problem (EIP1) *Determine the relative dielectric permittivity function $\varepsilon_r(x)$ in Ω for $x \in \Omega$ in nonconductive ($\sigma(x) = 0$) and nonmagnetic ($\mu_r = 1$) media when the measured function $g(x,t)$ s.t.*

$$\mathbf{E}\,(x,t) = g(x,t), \forall\,(x,t) \in \partial\Omega \times (0,T].$$

is known in Ω.

Inverse Problem (EIP2) *Determine the functions $\epsilon(x), \sigma(x)$ in Ω for $x \in \Omega$ for $\mu_r \approx 1$ in water assuming that $g(x,t)$ is known in $\partial\Omega \times (0,T]$.*

CIPs for magnetic field

Similarly can be obtained Maxwell's equations for magnetic field $\mathbf{H} = (H_1, H_2, H_3)$. Let us consider system of Maxwell's equations in linear, isotropic, nondispersive, lossy materials with $\sigma = 0, \sigma^* = 0$:

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \mathbf{E} \tag{94}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \frac{1}{\varepsilon} \mathbf{J}_{source} \tag{95}$$

In this case we take time derivative in (94) and operator $\nabla \times$ in (95) to get:

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} = -\frac{1}{\mu} \frac{\partial}{\partial t} \nabla \times \mathbf{E}, \tag{96}$$

$$\nabla \times \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} - \nabla \times \frac{1}{\varepsilon} \mathbf{J}_{source}. \tag{97}$$

Substitude the right hand side of (96) into (97) instead of $\frac{\partial}{\partial t} \nabla \times \mathbf{E}$ to obtain Maxwell's equations for magnetic field $\mathbf{H} = (H_1, H_2, H_3)$:

$$\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} + \nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} = \nabla \times \frac{1}{\varepsilon} \mathbf{J}_{source} \tag{98}$$

CIPs for for magnetic field

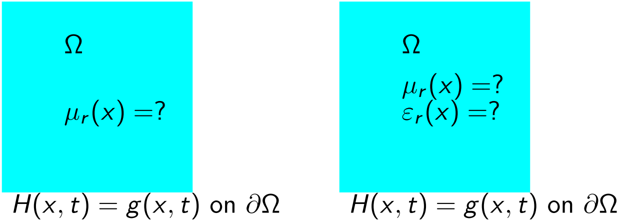
Let us consider now Cauchy problem for magnetic field $\mathbf{H}$ in the domain $\Omega_T = \Omega \times [0, T]$:

$$\begin{aligned} \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} + \nabla \times \frac{1}{\varepsilon} \nabla \times \mathbf{H} &= \nabla \times \frac{1}{\varepsilon} \mathbf{J}_{source} \text{ in } \ \Omega_T, \\ \mathbf{H}(\mathbf{x}, \mathbf{0}) = f_0(x), \quad \mathbf{H_t}(\mathbf{x}, \mathbf{0}) &= f_1(x) \text{ in } \ \Omega, \end{aligned} \tag{99}$$

- Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$ and specify time variable $t \in [0, T]$. Next, we supply the Cauchy problem by the appropriate b.c.
- In (99), $\varepsilon(x) = \varepsilon_r(x)\varepsilon_0, \mu = \mu_r \mu_0$ are dielectric permittivity and permeability functions, respectively, ε_0, μ_0 are dielectric permittivity and permeability of free space, respectively.

Different CIPs for time-dependent equation for magnetic field (99) can be formulated.

CIPs for magnetic wave propagation



Inverse Problem (MIP1) *Determine the relative magnetic permeability function* $\mu_r(x)$ *in* Ω *for* $x \in \Omega$ *in nonconductive* ($\sigma(x) = 0$) *media when the measured function* $g(x,t)$ *s.t.*

$$\mathbf{H}\,(x,t) = g(x,t), \forall\,(x,t) \in \partial\Omega \times (0,T].$$

is known in Ω.

Inverse Problem (MIP2) *Determine the functions* $\epsilon_r(x), \mu_r(x)$ *in* Ω *for* $x \in \Omega$ *assuming that* $g(x,t)$ *is known in* $\partial\Omega \times (0,T]$.

Maxwell's equations in 2D in a waveguide. TE and TM modes.

Let us assume that the structure being modeled extends to infinity in the z-direction with no change in the shape or position of its transverse cross section (case of a waveguide). If the incident wave is also uniform in the z-direction, then all partial derivatives of the fields with respect to z must equal zero. Under these conditions, the full set of Maxwell's curl equations given by (75) and (76) reduces to

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left[-\frac{\partial E_z}{\partial y} - \left(M_{source_x} + \sigma^* H_x\right)\right] \tag{100a}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z}{\partial x} - \left(M_{source_y} + \sigma^* H_y\right)\right] \tag{100b}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \left(M_{source_z} + \sigma^* H_z\right)\right] \tag{100c}$$

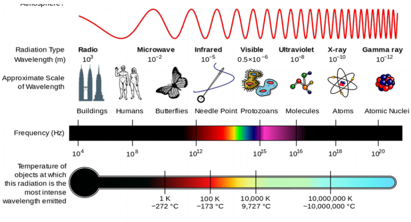
$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_z}{\partial y} - \left(J_{source_x} + \sigma E_x\right)\right] \tag{101a}$$

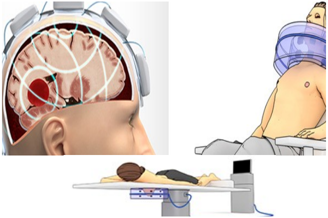
$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left[-\frac{\partial H_z}{\partial x} - \left(J_{source_y} + \sigma E_y\right)\right] \tag{101b}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \left(J_{source_z} + \sigma E_z\right)\right] \tag{101c}$$

If we will group (100a), (100b), and (101c), which involve only H_x, H_y, and E_z then we will set of field components to the *transverse-magnetic mode with respect to z* (TM_z) in two dimensions.
If we will group (101a), (101b), and (100c), which involve only E_x, E_y, and H_z. We shall designate this set of field components to the *transverse-electric mode with respect to z* (TE_z) in two dimensions.

Applications of CIPs for electric wave propagation





Left fig.: the electromagnetic spectrum (Wikipedia). Right fig.:Biomedical Microwave Imaging (frequencies around 1 GHz= 10^9 Hz) at the Department of Electrical Engineering at CTH, Chalmers, Göteborg, Sweden. Setup of Stroke Finder; microwave hyperthermia in cancer treatment and breast cancer detection, `https://www.chalmers.se/en/departments/e2/research/Signal-processing-and-Biomedical-engineering/`

Detection of explosives and airport security (usually X-ray technique)
`https://www.rsdynamics.com/products/explosives-detectors/miniexplonix/`