MATHEMATICS, Chalmers and Göteborg University
Financial Derivative and Stochastic Analysis (CTH-TMA285) (GU-MAM695)
December 16 2004, 14.00 to 18.00 at *Väg och vatten*.

SOME SHORT INDICATIONS OF SOLUTIONS.

---

1. (3p) Let Ω be the sample space of a dice toss.

   (a) Give the smallest, and the largest, σ–algebra in Ω, respectively. How many elements are there in those σ-algebras? Show also explicitly that these two sets fulfills the conditions for being a σ-algebra.

   (b) Show that the intersection of two σ-algebras is a σ-algebra.

   (c) Find an example of two σ-algebras whose union is *not* a σ-algebra.

   Solution: We have that $\Omega = \{1, 2, 3, 4, 5, 6\}$.

   (a) The smallest σ-algebra is the trivial one: $\{\emptyset, \Omega\}$, and the largest is the set of all subsets, i.e. the power set to Ω. This set has $2^6 = 64$ elements. It is easy to see that the unions and complements are included in those σ-algebras.

   (b) We have that complements and the unions in the intersection are also included in the intersection.

   (c) Let
   $$\sigma_1 = \big\{\{1\}, \{2, 3, 4, 5, 6\}, \emptyset, \Omega\big\}$$
   and
   $$\sigma_2 = \big\{\{1, 2\}, \{3, 4, 5, 6\}, \emptyset, \Omega\big\}.$$
   Then we have that
   $$\{1, 3, 4, 5, 6\} \in \sigma_1 \cup \sigma_2,$$
   but the complement to this set,
   $$\{2\} \notin \sigma_1 \cup \sigma_2.$$

2. (3p) The dynamics of $X(t)$ is given by the following stochastic differential equation for $t > 0$:
   $$t\, dX(t) = -\big(3 - \frac{4}{2 + t^2}\big) X(t)\, dt + t\beta(t)\, dW(t).$$
   Give $\beta(t)$ such that the variance
   $$\text{Var}\big(X(t)\big) = \frac{t}{(2 + t^2)^2}.$$

   Solution: We rewrite the SDE by divide by $t(\neq 0)$ and simplify to
   $$dX(t) = -\frac{2 + 3t^2}{2t + t^3} X(t)\, dt + \beta(t)\, dW(t).$$
   This is then reformulated to
   $$d\big(X(t)(2t + t^3)\big) = (2t + t^3)\beta(t)\, dW(t), \text{ that is,}$$

$$X(t) = \frac{1}{2t + t^3} \int_0^t (2u + u^3)\beta(u)\, dW(u) + \frac{1C}{2t + t^3}.$$

Which gives us that

$$\operatorname{Var}\big(X(t)\big) = E[X^2(t)] = \frac{1}{(2t + t^3)^2} \int_0^t (2u + u^3)^2 \beta(u)^2\, du.$$

In order to have

$$\operatorname{Var}\big(X(t)\big) = \frac{t}{(2 + t^2)^2}$$

we want that

$$\int_0^t (2u + u^3)^2 \beta(u)^2\, du = t^3.$$

This is obtained by choosing

$$\beta(u) = \frac{\sqrt{3}}{2 + u^2}.$$

3. (3p) Let $W(t)$ be a Brownian motion, and let $\Theta(t)$ be an adapted process to the filtration $\mathcal{F}(t)$ of $W(t)$. Let

$$Z(t) = \exp\big(-\int_0^t \Theta(u)\, dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)\, du\big), \text{ and let}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\, du.$$

Compute the differential $d(\frac{1}{Z(t)})$, and express the answer using the differential $d\widetilde{W}(t)$.

Solution: (See exercise 5.5.) We have that Itô-Doeblin first gives us that$dZ(t) = -\Theta(t)Z(t)\, dW(t)$, and then also

$$d(\frac{1}{Z(t)}) = -\frac{dZ(t)}{Z^2(t)} + \frac{dZ(t)\, dZ(t)}{Z^3(t)},$$

where we used the auxiliary function $f(x) = \frac{1}{x}$. This gives us that

$$d(\frac{1}{Z(t)}) = \frac{\Theta(t)\, dW(t)}{Z(t)} + \frac{\Theta^2(t)\, dt}{Z(t)} = \frac{\Theta(t)}{Z(t)} d\widetilde{W}(t).$$

4. (3p) Let us study a financial model with one stock and one bond. The value of the stock is given by $S(t) = S(0)e^{\alpha t + \sigma W(t)}$, and the value of the bond is $B(t) = B(0)e^{rt}$, where $W(t)$ is a normalized Wienerprocess. Let $\lambda = \frac{\alpha + \sigma^2/2 - r}{\sigma}$ and let $W^\lambda(t) = W(t) + \lambda t$.

   (a) Give a (financial) interpretation of λ.

   (b) We are interested of the stock's price process in relation to the bond, that is the process $R(t) = S(t)/B(t)$. Give an expression of the differential $dR(t)$ using the differential $dW^\lambda(t)$.

Solution: λ is usually called the market-price of risk and describes the normalized (division with σ) difference between the "drift" $\alpha + \sigma^2/2$ of the stock value,and the interest rate r of the bond.

Note that we can write $dS(t) = S(t)\big((\alpha + \sigma^2/2)\,dt + \sigma\,dW(t)\big)$ which we use in the solution below.

$$dR(t) = d(S(t)/B(t)) = d(S(t)e^{-rt}) = \text{ (Itô-Doeblin) } =$$
$$= S(t)((\alpha + \sigma^2/2) + \sigma\,dW)e^{-rt} - re^{-rt}S(t)\,dt = \ldots = \sigma S(t)e^{-rt}dW^{\lambda}(t) = \sigma R(t)dW^{\lambda}(t).$$

5. (3p) Let $K(t)$ be a certain exchange rate at time t. We model this rate by using a given volatility in the model
$$K(t) = C_0 e^{\int_0^t \sigma(u)\,dW(u)},$$
where $C_0 > 0$ and $\sigma(u) \in L^2[0,T]$. Let $k > C_0$ be a given critical level for this exchange rate. What is the probability, in the given model, that the exchange rate will stay below this level k, from $t = 0$ to $t = T$?

Solution: Let $x = \ln(k/C_0)$, then the probability is:
$$P[\max_{0 \le t \le T} \int_0^t \sigma(u)\,dW(u) \le x].$$
Using the reflection principle and that
$$\int_0^t \sigma(u)\,dW(u) \text{ is equal in distribution to } W\big(\int_0^t \sigma^2(u)\,du\big)$$
by Theorem 4.4.9 in Shreve's, we will get after some calculation that the probability for staying below the critical level is
$$\Phi\left(\frac{\ln(k/C_0)}{\int_0^T \sigma^2(u)\,du}\right).$$

6. (4p) (Vasičeks model)

   (a) State Vasičeks model for the interest process $R(t)$ on differential form, i.e. give the expression for $dR(t)$.

   (b) Derive the distribution of $R(t)$ in this model using Itô–Doeblin's formula, and the result (Theorem) on Itô integrals of a deterministic integrand.

   Solution: See Example 4.4.10 on p. 150 in Shreve's.

7. (4p) (Feynman-Kac)
State and prove the Feynman-Kac Theorem. Furthermore, make a short comment on the practical use of this theorem for financial applications.

   Solution: See Theorem 6.4.1 on page 268 in Shreve's book.